Nonharmonic Forcing of an Inverted Pendulum

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Abstract

The harmonically driven pendulum has been investigated thoroughly over the course of the past century. It is well known that the inverted state can be stabilized by appropriate harmonic forcing. We use the Jacobi elliptic functions to drive a pendulum and study the effects of nonharmonicity on the stability of the inverted state. By analyzing the problem numerically and experimentally, we conclude that the shape of the forcing function does not play an important role in determining the pendulum’s stability.

Keywords: inverted pendulum, nonharmonic forcing, Jacobi elliptic functions

1. Introduction

1.1. Motivation

The simple pendulum is one of the most fundamental problems in classical mechanics. When one adds a periodic vertical forcing to the pendulum’s pivot, rich dynamics emerge. One can achieve interesting dynamic behaviors including

- chaotic motion
- stabilization of the inverted state
- stable limit cycles about the inverted state

by simply choosing the correct parameters. Because of its unique combination of material simplicity and behavioral complexity, the pendulum provides the perfect system to study a broad class of phenomena in nonlinear dynamics.

1.2. Previous Work

The literature on this problem has a rich history dating back more than a century. Stephenson experimentally demonstrated the stability of the inverted, periodically-driven pendulum and gave an argument for its stability in 1908\cite{1}. Blackburn et al. investigated the parameter space of the problem more thoroughly using numerical techniques\cite{2}. This numerical research was complimented by an experimental study by Smith and Blackburn\cite{3}. In 1993, Acheson gave a theorem for the stability of the inverted N-pendulum\cite{4}. This stability was then demonstrated experimentally for N=1,2,3 by Acheson and Mullin\cite{5}. Sanjuán has explored the way that the shape of the periodic forcing term affects the pendulum using analytical and numerical techniques; his results indicate that there may be interesting behavior where the system can be switched from order to chaos to order by monotonically varying a single parameter of the external driving force\cite{6,7,8}

1.3. Equation of Motion

We are interested in the way that the shape of the forcing function affects the stability of the inverted state. Our pendulum shall be modeled with the following equation of motion:

\begin{equation}
I \frac{\partial^2 \theta}{\partial t^2} + b \frac{\partial \theta}{\partial t} - M r [g - A \omega^2 S(\omega t, m)] \sin \theta = 0 \quad (1)
\end{equation}

where $\theta$ is defined as the angle that the pendulum makes with the inverted state, $b$ is a damping coefficient, $M$ is the pendulum’s mass, $r$ is the radius of the pendulum’s center of mass, $g$ is the acceleration due to gravity, $A$ is the amplitude of displacement of the pendulum pivot, and $\omega$ is the pivot driving frequency in rad/s. $S(\omega t, m)$ is a stretched and scaled Jacobi elliptic function, which will be
defined in the following section. Note that this is the same equation used by Blackburn, [2], except for the substitution of $S(\omega t, m)$ for $\cos \omega t$.

In order to nondimensionalize this equation, begin by making the substitution $\tau = \omega t$. The pendulum’s natural frequency is given by $\omega_0 = \sqrt{M r / I}$ rad/s; this allows us to define the relative frequency of the driving oscillation as $\Omega = \omega / \omega_0$. The coefficient $Q = I \omega_0 / b$ is related to damping, and $\epsilon = \omega^2 A / g$ is the relative amplitude of the pivot’s displacement, and $\omega^2 A$ is the pivot’s peak acceleration. With these substitutions, the dimensionless equation of motion can be written

$$\ddot{\theta} + \frac{1}{Q \Omega^2} \dot{\theta} - \left[ \frac{1}{Q^2} - \epsilon S(\tau, m) \right] \sin \theta = 0. \quad (2)$$

1.4. Jacobi Elliptic Functions

In this work we shall investigate forcing terms that have the shape of Jacobi elliptic functions. These functions are interesting because they arise naturally in the solution of the motion of a simple pendulum [7]. Our forcing term shall be defined as

$$S(t, m) = \rho(m) \, \text{cn} \left( \frac{T(m)}{2\pi}, t, m \right). \quad (3)$$

Starting with incomplete elliptic integral of the first kind

$$u = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \quad (4)$$

the elliptic function $\text{cn}$ is defined as $\text{cn}(u, m) = \cos \phi$. $\text{cn}(u, m)$ is periodic in $u$ with period,

$$T(m) = 4 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \geq 2\pi, \quad (5)$$

and unit amplitude. Finally we have added a scaling factor,

$$\rho(m) = \left[ \int_0^{\pi/2} \int_0^t \text{cn} \left( \frac{T(m) u}{2\pi}, m \right) d u \, d t \right]^{-1}. \quad (6)$$

This scaling serves to enforce unit amplitude on the vertical displacement of the pivot when its acceleration is given by $S(t, m)$. Thus $\text{cn}(t, m)$ produces unit-amplitude acceleration while $S(t, m)$ produces unit-amplitude displacement. When $m = 0$ we have the special case $\text{cn}(t, 0) = \cos t$. This function is thus a generalization of the cosine function with an added parameter, $0 \leq m < 1$. Note that $T(0) = 2\pi$ and $\rho(0) = 1$, so that $S(t, 0) = \cos t$ is also a generalization of the cosine function. Because of this, the model presented here reduces to that used by Blackburn [2] in the case $m = 0$. The acceleration and resultant displacement associated with the forcing $S(t, m)$ are plotted in Figures 1 and 2.

![Figure 1: Pivot acceleration for various values of $m$.](image1)

![Figure 2: Pivot displacement for various values of $m$.](image2)

2. Experimental Methods

2.1. Apparatus

Our experimental apparatus is pictured in Figure 3. A computer running a LabView software controller is used to output the desired acceleration wave form. The output of this controller is then run through an amplifier which drives the shaker.
The shaker is essentially a heavy-duty loudspeaker with the equivalent of a giant voice coil driving the load.

The pendulum is made out of aluminum and uses an ABEC-9 in-line-skate bearing. There is a close-up of the pendulum itself in Figure 4. The pendulum’s range of motion is limited to approximately $\pm \pi/2$ radians because we are primarily concerned with the stability of the inverted state.

Data was collected by means of a high speed camera (under an aluminum foil shield in Figure 3) at approximately 200 fps. Another computer was connected to the camera. This second computer runs a LabView point-tracking code which records the 2D position of the white dot painted near the end of the pendulum.

2.2. Data Collection

We began simply by stabilizing the pendulum at 26Hz. From this state, we perturbed the pendulum to watch it recover its stability. We hoped to use this data to determine the fixed parameters, $\omega_0$ and $Q$.

Next we began sweeping through driving frequencies, $(2\pi \omega)$ from 20Hz to 50Hz and through values of $m$ from 0 to 0.999. At each point, we would turn up the gain on the amplifier until either the inverted state was stable or until we had exceeded the limits of the linear bearing or of the amplifier. Assuming we were able to stabilize the pendulum, we would then slowly lower the gain on the amplifier until the pendulum became unstable. We would then record the minimum driving amplitude required for stability by taking measurements with the point-tracking software.

3. Numerical Methods

In order to simulate the pendulum numerically, we integrated Equation 2 with a 4th-order Runge-Kutta routine from Matlab. In order to find the boundary of stability numerically, we swept systematically through values of $\Omega$. For each value of $\Omega$ a binary search was used to narrow in on a value of $\epsilon$ lying on the edge of stability. In order to judge stability an initial condition $(\theta, \dot{\theta}) = (1e-6, -1e-6)$ was integrated for 1000 periods of the forcing function and checked for divergence.

4. Results

The first step to analyze the experimental results is to compute the parameters $\omega_0$ and $Q$. This is done by manually fitting the numerical parameters to match experiment. The values $Q = 11$ and $\omega_0 = 5.8s^{-1}$ gave the best match as can be seen in Figure 5. Note that the nature of the amplitude envelope found numerically differs qualitatively from what was observed experimentally. In Figure 6 can be seen plots of both numerical and experimental results for the boundary of stability. While the two data sets do not line up with one another, they both indicate that the value of the parameter, $m$, has little affect on the boundary of stability.
Figure 5: Best match between numerical and experimental trajectories. Numerical parameter $Q = 11, \omega_0 = 5.8s^{-1}$.

Figure 6: Stability diagram with numerical and experimental results. The region above the data points is experimentally stable, the region below the data points is experimentally unstable. The same relationship applies for the numerically generated curves. Not that curves and point with different values of $m$ are nearly indistinguishable.

5. Discussion and Conclusion

It seems clear that damping is not the appropriate way to model the dissipative force in the system. Preliminary work suggests that a simple friction model would better explain the behavior we have observed experimentally. That is, the second term in Equation 1 should be replaced by a term such as

$$f \text{signum} \left( \frac{d\theta}{dt} \right)$$  \hspace{1cm} (7)

There are possibly large sources of error in both the numerically computed and the experimentally determined boundaries of stability. First, the numerics. The method we used to search for the border of stability is not robust. The computed data for $m = 0.99$ shows a dip at $\Omega = 50$ that is not present in the other data. This is more likely a demonstration of a quirky algorithm than an exciting phenomenon.

On the other hand, our experimental methods leave room for error as well. It was very difficult to classify the pendulum as being either stable or unstable very near the boundary. Sometimes it would take minutes for instability to manifest itself conclusively. Sometimes the pendulum would seem to find a new equilibrium state slightly to the left of vertical. Occasionally we observed a bi-stability between this slightly tilted state and the vertical. When this occurred the pendulum would appear to be stable for periods of approximately 30 seconds after which it would wander around before restablilizing.

Blackburn et al. [2] give a boundary of linear stability that lies directly between our experimental data and our simulated pendulum. It is not hard to imagine that our experiments are not sensitive enough to instability (not waiting long enough for destabilization) while our computed results are over-sensitive to instability (no robust search of phase space).

Despite this disagreement, our data tells a consistent story. Both our experimental results and our numerical results suggest that the parameter $m$, has no effect on the boundary of stability. The Fourier spectra of the pivot’s acceleration and displacement in Figures 7 and 8 hint at a possible explanation. The acceleration spectra show odd harmonics for $m$ close to 1, but these are almost entirely absent in the displacement spectra. The reason for this is that there is effectively a low pass filter between acceleration and displacement. The first acceleration harmonic present at 3 times the fundamental frequency is reduced to 1/9th its magnitude in the displacement spectrum. Essentially $m$ has a large effect on the shape of the acceleration, but it has little effect on the shape of the displacement.

Figure 7: Fourier spectrum of pivot acceleration

Figure 8: Stability diagram with numerical and experimental results. The region above the data points is experimentally stable, the region below the data points is experimentally unstable. The same relationship applies for the numerically generated curves. Not that curves and point with different values of $m$ are nearly indistinguishable.
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References