Time to Synchronization for Systems of Metronomes

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Coupled oscillators that synchronize through interaction have proved to be a useful for modeling a number of systems particularly in biology. All these systems share similar equations matching dynamics to a system of mechanical metronomes supported on a movable platform. In our project we consider this mechanical system due to its accessibility and freedom from stochastic noise present in many biological systems. Our work looks at how the length of time of the transient behavior depends on the number of oscillators, N. We considered systems of two to nine metronomes. We developed our own analysis methods, which reveal an upward trend in the amount of time to synchronization as we increase the number of metronomes. We extended an existing model for a system of two metronomes for the general case and ran simulations. Our analysis methods are comparable and further improvements are highlighted for future work.

I. INTRODUCTION & BACKGROUND

Christiaan Huygens first observed phase-locking in a system of pendulums in 1665. His serendipitous observation occurred while he was laying sick in bed for hours in front of a pair of pendulum clocks. He noted that these clocks synchronized in antiphase, which led him to conduct a series of experiments where he determined the stability of the system to return to antiphase motion despite perturbation.

This phenomenon was largely ignored up until recently. Bennett et al validated Huygen's results in 2002[?] . Further studies considered a system of two metronomes and looked at the roll of damping in the system[?] . Part of the reason for this resurgence is the existence of synchronous behavior in biological systems. Mirollo and Strogatz emphasized this prevalence by relating synchronous dynamics to firefly flashing, pacemaker cell firing, and populations of menstruating women[?] . Recently synthetic biologists explored intercellular quorum sensing in systems outfitted with genetic oscillators that could one day "function as spatially dis-

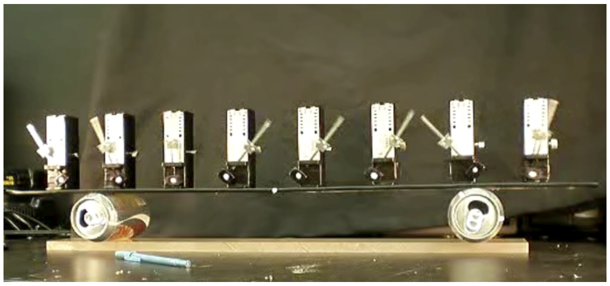

FIG. 1: Set-up. $N = 8$ case shown here.

tributed sensors or synthetic machinery for coupling complex dynamical processes across a multicellular population"[?] . Clearly there is a vast utility that extends beyond physics for understanding these systems.

Most work has revolved around modeling and understanding steady state behavior. We are interested in time-length of the transients or how long it takes for the system to synchronize, t_{sync}, if it does. We believe as biological models are utilized for synthetic purposes an understanding of the overall dynamics is necessary for optimization in the engineering. With this in mind we look at systems of metronomes to probe this question. Unlike biological systems, this mechanical system involves less noise and is experimentally viable within the given time-frame of a week. Nevertheless due to mathematical similarities of the dynamical equations, this work provides a necessary stepping stone before considering similar questions in less controllable contexts.

II. METHODS

A. Experimental Materials and Methods

The experimental set-up for this project is conveniently very simple, but care must be given to reduce external influences that can easily alter the dynamics of non-linear systems.

Our system consists of between two to nine metronomes, a flat, thick poster-board platform, and two rollers (empty 12 fl. oz diet coke cans). The basic set-up is shown in figure 1, with the metronomes always being placed along a 1-dimensional array. The metronomes are Wittners Super-Mini-Taktell (Series 880). We tested each metronomes capability of synchronizing in N=2 case. This resulted in us removing a metronome due to anomalous behavior. We tracked the data using a camera set for 42 frame per second that was connected to LabView on a PC. For data collection, we blacked out the pendulum surface and the reflective casing near the pendulum bob. This aided LabView in tracking a single white dot applied with white-out on each metronome bob. Similarly we spray-painted the poster-board platform black and applied a white dot on its side to track the motion. Finally we placed the entire system on a thin, smooth plastic block used to reduce frictional irregularities liable from the table surface.

During our week in the lab we ran 20 usable data-runs for $N = 2, 3, 5, 6, 7, 9$, 19 data-runs for $N = 4$, and 12 runs for $N = 8$. The metronomes were set for 200 beats per minute, which corresponds to 100 periods per minute. For each run we set up LabView to track the platform and the metronome bobs. Once the camera was tracking we stabilized the platform by hand as we set the metronomes one-by-one in motion. As the last metronome was set in motion we let go of the platform. We considered this the beginning of the run. The system was allowed to evolve for 90 seconds or longer depending on whether synchronization seemed to occur. For some of the higher N cases synchronization did not occur despite allowing the metronomes to wind completely down.

B. Experimental Analysis

Our goal was to determine the dependence of time to synchronization on the number of metronomes. We used the raw data output of times and x-positions (from the camera lens point of view in pixel units). We considered a strict definition of synchrony where metronomes are in-phase.

Due to camera distortions and small tracking errors, we chose to consider the turning points of the data as opposed to the entire time series. By using the findpeaks function in Matlab our time-series data was reduced to a vector of peak times for each metronome. We chose to look at peaks occurring at only one turning point ensuring our analysis would not falsely detect anti-phase phase-locking as an occurrence of our stricter synchronization.

We developed an algorithm for determining synchrony for use with the experimental data and the simulation data. The peak times will occur at the same time for systems in inphase synchrony. However for the real data, this is not expected even when synchronization has phenomenologically occurred because of slight noise in the system, errors in tracking, and discretization of time. As a result we cannot discount synchronization from being achieved even if peaks occur at slightly different times for the metronomes. This motivates a time-ϵ definition of synchronization. We introduce the model parameter ϵ_{time} where if all metronome peaks occur within it we consider that time an instant of synchrony. Full synchronization occurs when so many instances of synchrony occur in a row (defined by another model parameter, r).

After trimming the data of all time before the platform is destabilized and determining the peaks, the output data is reduced to a vector of peak times for each metronome. The size of each vector is not necessarily the same, however any instance of synchrony must contain any elements from each vector so we look for instances of synchrony by arbitrarily choosing one vector and considering its time elements in succession, v_{arb}. In order to reduce the arbitrariness we used an algorithm that updated the reference time for placing the ϵ_{time} ball. Initially we look around the arbitrary metronome peak time. Any peak times from other metronomes within the ϵ_{time} are placed in a set with the original reference peak time and the average time becomes the new reference for the ϵ_{time}. This procedure is continued until no further peak times are picked up. At this point, if the number of elements is equal to N we consider it a syncing instance and place the reference time in a set, $\mathcal{S}$, and we move to the next peak time in v_{arb}. If an instance of syncing does not occur $\mathcal{S}$ becomes the empty set and we move to the next peak time in v_{arb}. When the cardinality of $\mathcal{S} = r$ we take the $min(\mathcal{S})$ as the time of synchronization, t_{sync} for the run. We acknowledge this method still contains an element of arbitrariness that can lead to errors. A minimal error method realized after our data analysis and presentation as a group is outlined in the conclusions.

C. Theory and Simulation

Part of modeling the system correctly requires an understanding of how mechanical metronomes operate. Key to their timekeeping is an escapement mechanism. As one winds up the metronome before use the spring inside is supplied with potential energy that is gradually used up by providing a kick (via a system of parts) to the rod of the pendulum. This recovers of the energy dissipated during every oscillation. As long as the metronome is wound it can maintain the same frequency. This physically manifests itself with the escapement catching the rod causing a slow down as it approaches the vertical and then an increase in speed through a kick as the rod passes the vertical. Aware of this we can accurately implement the equations of motion.

We generalize the equations of motion that have been laid out for the $N = 2$ case[?] . For a system of N metronomes sharing a platform we have the equations of motion for the jth metronome rod and the platform center of mass:

$$\ddot{\phi}_j + b\dot{\phi}_j + \frac{g}{\ell}sin(\phi_j) = -\frac{1}{\ell}\ddot{X}cos(\phi_j) + F_j$$

$$(M + Nm)\ddot{X} + B\dot{X} =$$
$$-m\ell(sin(\phi_1) + sin(\phi_2) + \ldots + sin(\phi_N))^{\cdot\cdot}$$

where the last double dot is for the entire right hand side in the second equation. ϕ_j

refers to the angular displacement of the jth metronome rod, X is the platform center of mass linear displacement, b is the viscous damping term, g is acceleration due to terrestrial gravity, ℓ is the effective rod length of the metronome, F is the impulsive drive, M is the platform mass, m is the mass of the metronome rod and weight, B is the platform friction coefficient and dots denote derivatives with respect to time t. We can non-dimensionalize by introducing scaled variables $Y = X/\ell$ and $\tau = t\sqrt{g/\ell}$. Thus we can write:

$$\phi_j^{''} + 2\tilde{\gamma}\phi_j^{'} + sin(\phi_j) = -Y^{''}cos(\phi_j) + \tilde{F}_j$$

$$Y^{''} + 2\Gamma Y^{'} =$$

$$-\mu(sin(\phi_1) + sin(\phi_2) + \ldots + sin(\phi_N))^{''},$$

where prime refers to differentiation with respect to τ and we have defined

$$\mu = \frac{m}{M + Nm},$$

$$\tilde{\gamma} = b\sqrt{\frac{\ell}{4g}},$$

$$\Gamma = \frac{B}{M + Nm}\sqrt{\frac{\ell}{4g}}.$$

For the escapement mechanism we incorporated a transformation for whenever a $\phi_j = 0$. For the angular velocity $|\phi_j^{'}| \rightarrow \gamma|\phi_j^{'}| + c$ where $\gamma < 1$ is the factor reduction in the speed as the escapement catches and c is a fixed impulse kick in the direction of motion. For our model the kick affects the amplitude but not the phase.

We simulated our model in Matlab using $ode45$ with event recognition to incorporate the escapement mechanism physics and random initial conditions. Ode45 numerically integrates systems of equations using a Runge Kutta (4,5) method. We ran 100 runs For each N case. We determined the time to synchrony using the same method outlined for the experimental data for comparison. We also used a "strict sync test" where we defined a synchronous event to occur if all metronome peaks occurred within two time elements. The rationale behind this was that there is less noise with the data so that a synchronous event occurs if all peaks occur in a column along the time axis. Machine precision will alter this column slightly and discretization of time will cause all points to reside in either one of two possible time locations. Simply put, we know a synchronous event does not occur if the element distance between any two peaks is two or more. For one or no peak element differences, we can not conclusively say whether a synchronous event occurs or not. So we assume that a synchronous event does occur and rely on the value of r to correct for synchronous event identification errors. This strict test should give a more accurate t_{sync} available only to noiseless simulation results and provides an

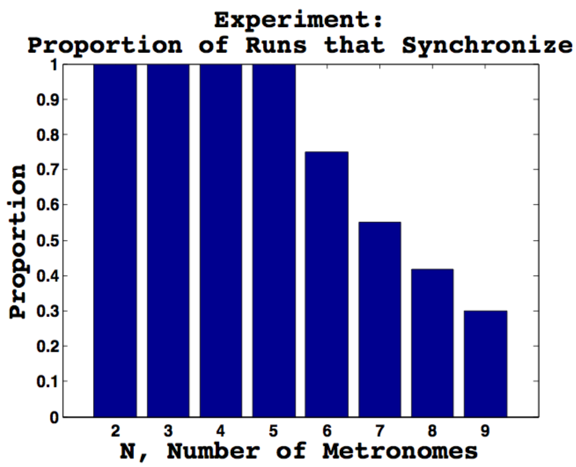

FIG. 2

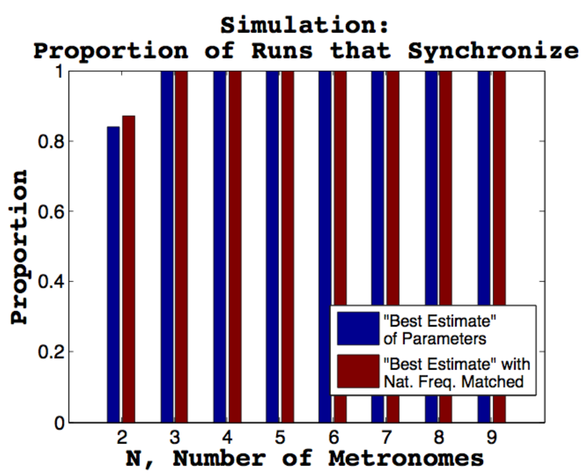

FIG. 3: Best estimate parameters: $M = 0.0655 + N(.094 - m)kg, m = .022kg, b = 0.0022, B = 0.001, \gamma = 0.97, c = .025, \ell = 0.025m$

Matched freq: Same as above but $\ell = 0.1m$

upperbound on any experimental results.

III. RESULTS

As outlined, we determined t_{sync} for each data run. For the simulation we considered two different parameter sets. One we measured what parameter values we could and gave a best guess for other parameters. We noticed that the final frequency for this simulation did not match experimental results. For our second parameter set we matched the frequency by taking the old parameter set and scaling the lengths of the metronome rods, ℓ. Our values are captioned in figure 3.

Figure 2 shows the percentage of runs for each N that synchronize in the experiment. We observed synchronization for all the runs for $N \leq 5$ and a decrease in occurrence of synchronization as we increase N. Figure 3 shows the same graph but for the simulation. Figures 4 and 5 show the average t_{sync} for each N with error bars of the standard deviation for the set of synchronized runs for each N. Both figures show the results data but figure 4 is using our "best guess" parameter values and figure 5 is after rescaling ℓ to match the frequency. We note there is a slight upward trend and larger uncertainties as we deal with smaller sets of runs for high values of N. Our simulations match experimental results especially when we corrected for the frequency.

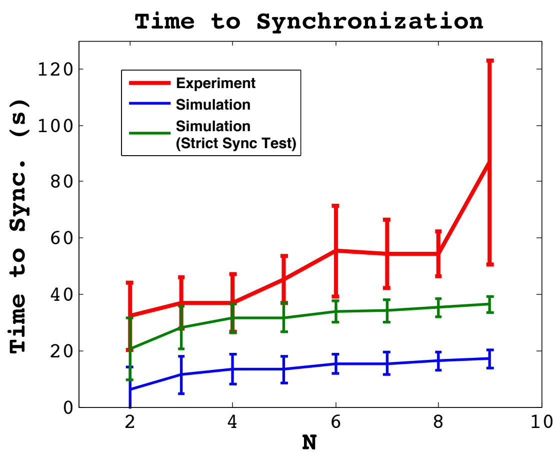

FIG. 4: Means of runs that synchronized in each data set with standard deviation error bars.

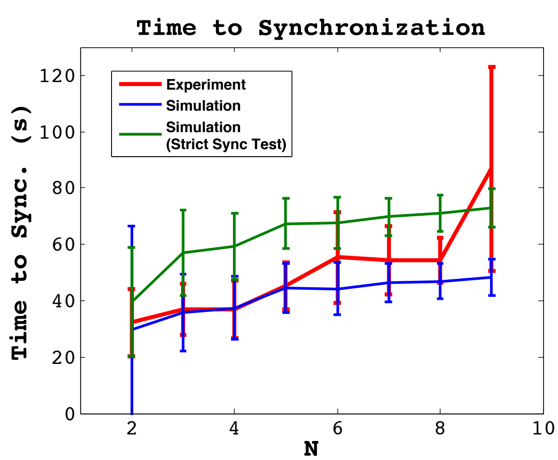

FIG. 5: Means of runs that synchronized in each data set with standard deviation error bars

IV. DISCUSSION

Our results show that when using the same method of analysis on both experiment and simulation we have agreement between basic statistical measures. This suggests our two schema are comparable. At higher N values there is less agreement. A methodological source for differences is that we have very few data samples for the high N values. For example, for $N = 9$ only six runs resulted in synchronized behavior. A simple solution would be take more data. Another source of error is the fact that as we introduce more components in a system each component introduces its own internal source of error both mechanistically and theoretically–clearly the model does not account for all slight contributions to the dynamics and parameters related to each metronome do not perfectly match reality. Together this suggests that simulation and experiment should diverge as we increase the value of N.

There are a number of sources for methodological error that could have been better minimized. After our presentation Professor Schatz suggested that instead of just looking at given peaks we could more accurately determine peak times by determining a weighted average of all the data points in each half-period. The theoretical challenge in this is how exactly to weight and what range around the given peaks should be considered. If implemented correctly this should give more accurate peak times. After analysis we realized a time-ϵ method that completely eliminates the arbitrariness of the utilized algorithm. Here we would still use a reference metronome to determine plausible times for instances of synchrony. Now we would look around the reference point by twice ϵ and put all metronome peak times that exist within this time neighborhood into a set, $\mathcal{S}$. If the number of elements $\mathcal{S}$ is equal to N and $max(\mathcal{S}) - min(\mathcal{S}) \leq \epsilon$.

The greatest source of error however is that while we're interested in perfect synchronization the time-ϵ method incorrectly identifies phase locking with very small phase differences as perfect synchrony. Hindsight suggests that instead of dealing with a time-ϵ for peak times a more complete analysis is to deal with a total-phase-difference-change-ϵ. The method is similar but requires that we have a means of obtaining angular displacement accurately. Given the difficulty and inaccuracy of measuring metronome rod length it is not clear we can obtain it from the raw data we have. For illustrative purposes during our presentation we used the Hilbert Transform to deduce the angular displacement from our data, however this transform introduces errors and would not be useful with methods that look so locally at the data. The simplest solution is to track an extra dot on each metronome placed at the metronome rod pivot point and use simple trigonometry to obtain the phase values. Then we would look at the time-series of the sum of pairwise absolute value of differences in angular displacement. We would look at the difference between adjacent time points (akin to the time derivative) and when this value is less than some ϵ for some total time amount (analogous to r) we would consider the system to be phase locked. The final phase locked value of total-absolute-phase-difference is then a measure of how anti-phase the system ended up. This final metric would require some further thought to correctly account for counting differences between each N case.

V. CONCLUSION

Our work is still useful despite the mentioned downfalls of our methods. The proposed updated methods are refinements and the results we obtained should still be qualitatively accurate. Indeed our experimental results match intuition of an increasing time to synchrony with N and a decreasing percentage of synced cases with N. Our simulation results match experiment, which suggests that the model is a reasonable comparison. Some differences occur between experiment and simulation. First of the simulation always synced up for all N cases except for the $N = 2$ case, which always synced up in experiment. We believe the source of this discrepancy is the choice of parameter values. As N is a parameter in the model, we believe at low values of N our system is on the cusp between in-phase and anti-phase stable steady states. We postulate that if we increase simulation time phase-locking may be observed or that both steady states are possible depending on initial conditions. This is counter to what we observed so a more accurate parameter set likely exists.

Overall our analysis shows that synchronization times have a very slight upward trend as N increases whereas likelihood of synchronization at a certain point vastly drops off as N increases. When synchronization did not occur there was still an heuristically observed locking of the system with multiple frequencies. Future work could analyze how phase-locking occurs within subsets of the metronomes. Additionally future work should heed to our suggested more precise methods.

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