The Completely Inelastic Ball: Analysis and Experiment

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We consider the problem of a perfectly inelastic bouncing ball on a vibrating plate subjected to a superposition based input forcing function. Both, analytical and experimental studies are performed and the results from each are compared. The developed theoretical models also incorporate friction and provide a fair agreement with the actual experimental data. Interesting regions on bifurcations diagrams are identified validated from the actual data. We also generate return maps from the experimental data, depicting single, double and bi-modal behavior of the dynamical system.

I. INTRODUCTION

Bouncing ball on a vibrating plate has been a subject of active research for over the last few decades[5,8]. The experiment beautifully demonstrates the main theory of chaos and bifurcation while still being easy enough to validate experimentally. Hence, over the years numerous variations of this problem ranging from bouncing objects[2] to varying geometry[3] have surfaced. These variations have practical application in areas like design of vibration dampers[6] and study of coordination in humans[7]. In particular, the case of completely inelastic bouncing ball has direct applications in the field of granular media physics[1].

The inelastic bouncing ball is a particularly interesting problem and has been a subject of many recent studies[4,5]. The problem for a simple harmonic vibrating platform has been partially solved by Luck and Mehta[5]. Their main assumptions on the flight time, forces the ball to always reach the sticking region. More recently, Gilet, Vandewalle and Dorbolo[4] have fully resolved the motion of the inelastic ball subjected to harmonic vibrating plate without these limiting flight time assumptions. Their study has demonstrated the presence of pseudo chaotic behaviors along with the presence of bifurcation cascades (infinite of them).

In this study we intend to build upon the insightful analysis done by Gilet and others and study the behavior of a completely inelastic bouncing ball on a vibrating plate subjected to a superposition based input forcing function. Apart from experimentally validating the case of single sine based forcing function, the present works also extends the mathematical analysis performed by Gilet by incorporating superposition of two sine waves with integer-multiple frequencies as the forcing function. One immediate effect of the superposition is the presence of time dependent takeoff constraint which leads to a dynamic bifurcation behavior. The results from a single sine based forcing function are found to be in good agreement with the experimental data when we take friction into account in our theoretical models.

We start by outlining the general mathematical model of a inelastic bouncing ball, followed by a brief overview of the experimental setup, the analysis of results and finally the conclusion.

II. MATHEMATICAL MODEL

The dynamical system understudy here is that of a inelastic ball subjected to sum of two sine waves based forcing function given by

$$x_p(t) = A(\sin(\omega t) + \sin(c\omega t + \theta)) \tag{1}$$

where A is the amplitude, ω is the frequency, c is the frequency ration between the two waves and θ is the phase shift between the waves. For the current study we keep the phase shift (θ) equal to zero. Subsequently, the dynamics of the systems are governed by the following two differential equations

$$\dot{x}_p(t) = A(\omega\cos(\omega t) + c\omega\cos(c\omega t + \theta)) \tag{2}$$

$$\ddot{x}_p(t) = -A(\omega^2\sin(\omega t) + c^2\omega^2\sin(c\omega t + \theta)) \tag{3}$$

The governing equations as made dimensionless by defining three non-dimensional parameters as follows

$$\varphi = \omega t \tag{4}$$

$$\Gamma = \frac{A\omega^2}{g} \tag{5}$$

$$\chi = \frac{x\omega^2}{g} \tag{6}$$

φ is a generalized time, Γ is a generalized acceleration and one of our control parameters, and χ is a generalized position. The other control parameters are the ratio of the frequencies c and the phase shift θ (currently set to 0) between the sine waves. Hence, the new dimensionless forcing function is now given by

$$\chi_p(t) = \Gamma(\sin(\varphi) + \sin(c\varphi + \theta)) \tag{7}$$

The ball takes off whenever the upward acceleration of the plate exceed the acceleration due to gravity(g), or in other words when the plate is not in contact with the ball. The mathematical condition which satisfies this constraint is given by Eq. 8 (Note $\theta = 0$)

$$\ddot{x}_p(t) = -A(\omega^2 \sin(\omega t) + c^2\omega^2 \sin(c\omega t)) \leq -g$$
$$-A\omega^2/g(\sin(\omega t) + c^2 \sin(c\omega t)) \leq -1$$
$$\Gamma(\sin(\varphi + c^2 \sin(c\varphi)) \leq -1 \tag{8}$$

It is expected that the ball will take off whenever the above condition is satisfied. For a single sinusoidal input function the take off condition simplifies to following

$$\Gamma(2\sin(\varphi)) \leq -1 \tag{9}$$

here we have applied the condition that $\theta = 0$. Further, we can analytically determine the position of the ball relative to the plate as follows

$$\begin{aligned} x_b(t) =& \Gamma(\sin(\varphi_0) + \sin(c\varphi_0) - \ldots \\ & \sin(\varphi)\sin(c\varphi) + \Gamma(\omega\cos(\varphi_0) + \ldots \\ & c\omega\cos(c\varphi_0))(\varphi - \varphi_0) - 1/2(\varphi - \varphi_0)^2 \end{aligned} \tag{10}$$

Finally, the flight time defined as $T = \varphi_i - \varphi_0$ can be found by solving the following equation numerically:

$$\begin{aligned} F \equiv& \Gamma[\sin(\varphi_0)(1 - \sin T) + \ldots \\ & \sin(c\varphi_0)(1 - \sin cT) - \cos T\cos(\varphi_0) - \ldots \\ & \cos(cT)\cos(c\varphi_0)] - \Gamma[\cos(\varphi_0)\ldots \\ & + c\cos(c\varphi_0)]T - 1/2T^2 = 0 \end{aligned} \tag{11}$$

A. Model Performance

Figure 1 shows the position diagram for $c = 1$ and $c = 2$. The first case ($c = 1$) corresponds to a single sine wave based forcing function and successfully reproduces the behavior discussed by Gilet et. al. It is observed

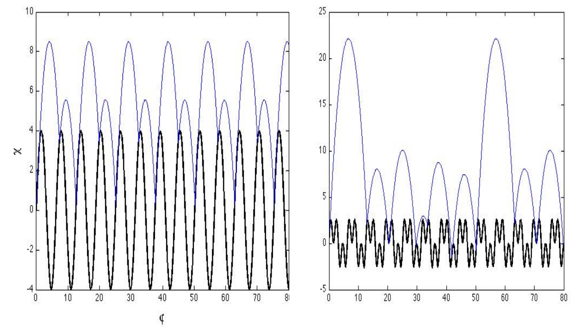

FIG. 1. Generalized position χ as a function of φ for a superposition of two sine waves with no phase shift ($\theta = 0$),$c = 1$ and 2, and $\Gamma = 2$ and 1.28. Black is the motion of the plate, and blue is the ball.

that the ball can land in two distinct regions. A 'bouncing' region, where the acceleration of the plate is such that the ball immediately takes off again, with an initial velocity equal to the instantaneous velocity of the plate at the time of impact. The other region is the 'sticking' region, where the ball is not able take off, and instead rests on the plate until the plate's acceleration is such that is can take off again. This region resets the time history of the ball and hence marks the boundary between periodic cycles.

By varying the control parameter Γ from 0 to 8, we also construct the bifurcation diagrams for the cases $c = 1, c = 2$ and $c = 3$, given by Fig. 2. For the $c = 1$ we can clearly see that there three main regions. 1) The bouncing region on defined by the left most boundary or values, 2)the sticking region depicted along the branches at bifurcation points and the forbidden region (the white spaces in between). For $c = 2$ and 3 the structre is quite complex but you can still differentiate out these three main regions for lower gamma values.

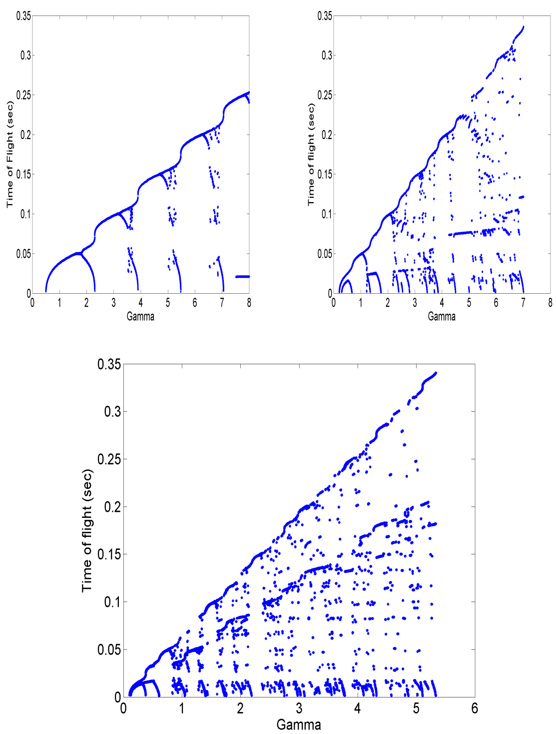

FIG. 2. Bifurcation plots for two sine waves with no phase shift ($\theta = 0$) and c = 1(left), 2(right) ,3(bottom). We can see the bouncing, sticking(branches) and forbidden region (white spaces)

Next we give a brief overview of the experimental setup used for collecting time of flight data.

III. EXPERIMENTAL SETUP

The inelastic bouncing ball experiment involves a horizontal shaking platform driven vertically according to a forcing function inputted by an oscilloscope or computer. A small, inelastic ball is placed on the plate and it's motion is recorded by a high speed camera. To obtain the time of flight data there were two setups which were considered. The first setup involves using the accelerometer to detect the ball's impact time with the shaker plate. This configuration suffers from various drawbacks like noisy signal due to soft impacts and deviation of ball from its 1d motion path.

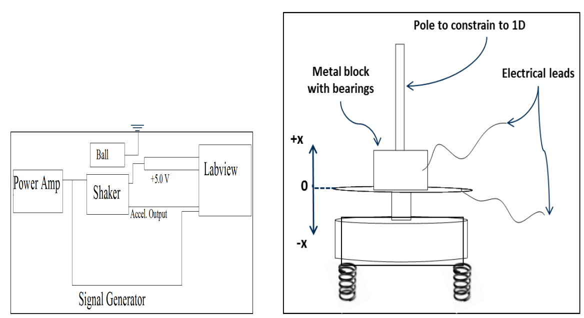

FIG. 3. Impact detection circuit and General setup

To overcome these problems a new experimental setup has been devised. The new setup(shown in Fig. III consist of a metal block (mass=249g) with ball bearings was attached to a vertical pole to stabilize the motion in 1-D. Furthermore, to generate accurate time of flight data, an impact detection electrical circuit is setup (Fig. III). The impacts correspond to the current being short circuited. Figure 4 shows the output data from this circuit. The ball is in flight when 5 V voltage is registered. We can see that this setup provides a near bang-bang impact detection with little or no noise.

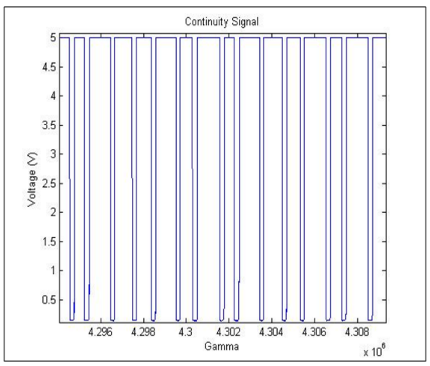

FIG. 4. Impact signal voltage

A. Incorporating friction

During the course of the experiment it was realized that friction plays an important role in the dynamics and hence has been incorporated in the final mathematical model. To measure the columbic friction between the wall and the vertical post, a vertical timed drop test was performed and a resulting parabola was fitted to calculate the final friction value of 0.3g. This friction was incorporated in our mathematical model with a simple change to the equation of motion of the ball relative to the plate, as below

$$\begin{aligned} x_b(t) =&\Gamma(\sin(\varphi_0) + \sin(c\varphi_0) - \dots \\ &\sin(\varphi)\sin(c\varphi) + \Gamma(\omega\cos(\varphi_0) + \dots \\ &c\omega\cos(c\varphi_0))(\varphi - \varphi_0) - 1/2(\varphi - \varphi_0)^2 + \dots \\ &+ 0.3/2(\varphi - \varphi_0)^2 \frac{Vel_{bp}}{|Vel_{bp}|} \end{aligned} \tag{12}$$

where Vel_{bp} is the velocity of ball with relative to plate. We note that friction may increase or decrease the time of flight as it acts on the relative and not the absolute velocity of the ball. In general, it is found that incorporating friction shifts the bifurcation diagram to the left.

B. Data acquisition

The data has been taken at a sample rate of 20 KHz. Labview is used to read in and process the data. Two channels are read in to LabView, 1) acceleration and 2) voltage from continuity sensor. This data is then parsed and stored in tab delimited format to text files with two additional columns for time and acceleration due to the impact of the ball. We note that the experimental setup was only capable of providing single sin wave input forcing function.

The actual experiment consists of ramping up the amplitude of vibration to generate accelerations from 1g to 8g, with frequent wait periods in between two defined amplitude values. The number of periods to wait before increasing the amplitude is controlled by the user. Waiting periods are varied from 5 to 60 and selected according to the quality and size considerations of collected data. For generating time of flight vs Γ bifurcation diagram , we first use matlab's reshape command to reshape the data such that each step of constant amplitude maybe be analyzed independently. The TOF values are then determined for each constant amplitude group with the first few TOF values being ignored to capture the steady state response of the system. Plotting the TOF values for each of the amplitude groups gives the required bifurcation diagram. Figure 5 shows the final experimental setup.

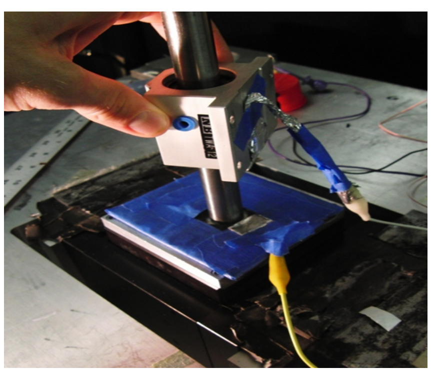

FIG. 5. In lab setup

IV. ANALYSIS AND DISCUSSION

Figure IV shows the predicted bifurcation (with friction)and the actual bifurcation plot generated from experimental data respectively. We also show the predicted bifurcation diagram, with friction for the case of $c = 3$.

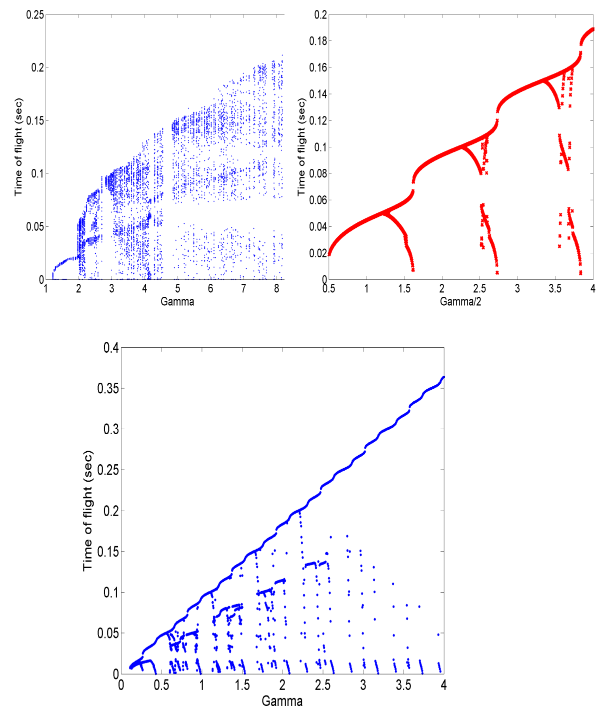

FIG. 6. Bifurcation diagram, c = 1: Predicted (right) vs Actual; C = 3 Predicted (bottom)

The Γ values for the predicted bifurcation plot should be doubled when comparing to the actual experimental bifurcation diagram. On inspection we can see that our model performs fairly well and is able to predict the general structure and the time flight flight values to a fair degree of accuracy. We can also observe the three regions in our experimental data, especially for lower values of Γ where the data is more clean.

To further investigate and visualize hidden patterns, return diagrams (N+1 diagram) have also been plotted for each section of the bifurcation diagram. Figure 7 shows three of these diagrams depicting single bounce mode($\Gamma = 1.75$), two bounce mode (Γ =2.25), and the bimodal pattern mode(Γ = 3.5g)

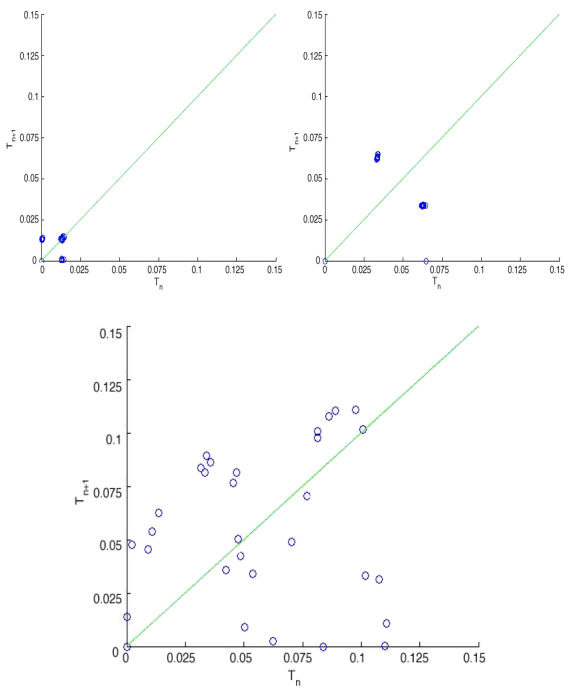

FIG. 7. Return diagrams showing period doubling bifurcations: Single(top left) , Double (right) and bimodal (bottom) modes

V. CONCLUSION

In this research we have successfully studied the behavior of a completely inelastic bouncing ball on a vibrating plate subjected to a multiple sinusoidal based input forcing function. To improve the model accuracy, coulomb friction has also been incorporated in the theoretical model. Experimental impact data is acquired via a voltage based impact detection circuit. The raw voltage data is post processed in matlab to obtain the actual flight times of the bouncing ball. Bifurcation diagrams. both with the theoretical model and experimental data have been generated. The results, with friction included are a close match with the actual experimental data. Return or N+1 maps have also been generated, which show period doubling and bi-modal behavior of our dynamical system. The discrepancy between absolute TOF values maybe attributed to possible sources of error like plate idle feedback, torquing impact of the ball and elasticity of the bouncing ball. Future work may include improving the experimental setup to reduce the free vibrations in the shaker, incorporating multiple sine waves during the experiment and incorporating elasticity directly in the theoretical model.

VI. ACKNOWLEDGMENTS

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