The Time to Synchronization for N Coupled Metronomes

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The synchronization of coupled oscillations has proven to be a useful model for describing the dynamics of a large number of physical systems. Perhaps the simplest system which can capture such dynamics is that of two oscillators coupled through the motion of a free platform. In recent years, there has been a considerable amount of research on the case of $N = 2$; however, little work has extended to larger numbers of oscillators. The goal of the current work is to quantify the time to synchronization for $N = 2$ through $N = 9$ metronomes on a movable platform, through experimental investigation and numerical simulation. Our results show a considerable level of agreement between the two, and discrepancies are explored. Possibilites for future work and potential improvements are noted.

I. INTRODUCTION

The phenomenon of synchronization is found throughout nature and in many applications of different fields of science. Examples of synchronization[1] include the flashing of firefly populations, the firing of pacemaker cells in the heart, applications of Josephson junctions, coupled fiber laser arrays, and perhaps even epileptic seizures. Many instances of synchronization in nature and applications in science become extremely complicated as soon as multiple variables or several coupled oscillators are introduced. However, the pervasive presence of synchronization in nature and science motivates work towards a deep understanding of such phenomena. So, as with any system that is to be scientifically understood, the simplest case is the best place to begin; for synchronization, the case of two identical, coupled oscillators is perhaps the simplest.

Research into the mutual synchronization of two identical coupled oscillators is believed to date back to the work of Christiaan Huygens in the 1600's, in which he discovered and investigated the synchronization of two pendulum clocks hanging on a common support beam[2]. Motivated by the goal of solving the "Longitude Problem," Huygens reached some insightful conclusions about the nature of this system, eventually concluding that small movements of the support beam with every oscillation were responsible for the antiphase synchronization that consistently occurred.

Significant progress has been made in explaining simple instances of synchronization. In 2002, Bennett et al.[3] constructed an apparatus meant to recreate the setup described by Christiaan Huygens. The experimental and numerical results agree qualitatively with records from Huygens's research. Concurrently, Pantaleone[4] investigated a similar setup in which metronomes were placed on a common support which was free to roll across cylinders, allowing coupling through the low-friction horizontal translation of the platform. Pantaleone's theoretical analysis provided agrees with the experimental observations reported.

In 2009, Ulrichs et al.[5] reported agreement with Pantaleone's results in computer simulations. Additionally, these simulations were run for coupling of up to 100 metronomes and reportedly showed chaotic and hyperchaotic behavior. A paper by Wiesenfeld and Borrero-Echeverry[6], currently in the process of publication, investigates how varying the parameters of two coupled oscillators can affect the synchronization outcome.

One may argue that a strong theoretical understanding of the $N = 2$ case has been developed in recent years; it seems appropriate to expand this theory to a larger number of coupled oscillators. This is the goal of the current work.

II. EXPERIMENT

The experimental setup, similar to that of Pantaleone[4], is depicted in Figure 1 for the case of $N = 3$. For each value of N, 2 through 9, the corresponding number of metronomes (model: Wittners Super-Mini-Taktell Series 880) are placed side-by-side on a rigid styrofoam platform, which is then placed on top of two aluminum soda cans; the cans are placed upon a thin, smooth plastic block intended to reduce frictional effects. The orientation is such that the platform is free to one-dimensional translation in the same plane as the pendulum oscillations. It is these oscillations of the platform which provides the coupling of the metronomes.

The metronomes are started with "random" initial conditions, by sequentially tilting each pendulum to about 45° and then releasing it. During this time, the platform is held steady; with the release of the Nth pendulum, the platform is released as well, meaning that the metronome coupling begins at this time. The entire experimental setup is recorded with a video camera, which streams live into a computer via a firewire. Labview software is used to track a white dot that has been placed on each pendulum bob and the center of the platform. This setup is depicted in Figure 1, and an actual image of the setup is provided in Figure 2. The resulting data is a text file containing a series of times and the x and y positions of $N + 1$ points.

This data is then read into Matlab for analysis. We make use of only the x coordinates, since the largest displacement is in that direction. For illustrative purposes, the x displacement in pixel units as a function of time is plotted in Figure 3. To shift all these oscillations to a common origin, we subtract each metronome's mean x-value from its time series. This effectively shifts each metronome's oscillations to be about zero. The beginning of a data run for $N = 2$, after applying this shifting, is depicted in Figure 4 (Top).

Now, using Matlab's "findpeaks" function, we are able to determine the time to synchronization. The first step is to determine the time at which coupling begins. This value, as previously described, will be the first peak (in absolute value) of the Nth metronome. This is indicated by a red circle in Figure 4 (Top). To then determine the time to synchronization from this point, we use a quick and simple method. Synchronization, by definition, refers to oscillations in unison; when the peaks of the time series of x-positions align, we have this unison, and thus synchronization. A complication to this matter, however, is the noise and experimental uncertainty inherent to physical measurements, which makes the peak value that is returned not necessarily match the center of the oscillation, as illustrated in Figure 4 (Bottom).

To quickly and simply remedy this issue, we do the following. For the first peak of an arbitrarily chosen metronome, we look some ϵ away in both directions and note the location of every other metronome's peak within that range. We then average the time values of those peaks to obtain a better estimate of the center of the group of local maxima of each metronome. From this new value, we look ϵ away in each direction and test for N metronomes. If all N are found, we test that this condition is satisfied for the next sequential k peaks. If so, this time marks synchronization; otherwise, we step forward in time and test again.

To determine what value of ϵ to use, we note that too large a value will result in too lenient of a test for synchronization, but too small of a value will not account for the peak uncertainty as depicted in Figure 4 (Bottom). Additionally, the time discretization of the firewire image streaming requires $\epsilon > 20ms$. We determine that $\epsilon = 40ms$ and $k = 20$ give reliable results for measuring the time to synchronization.

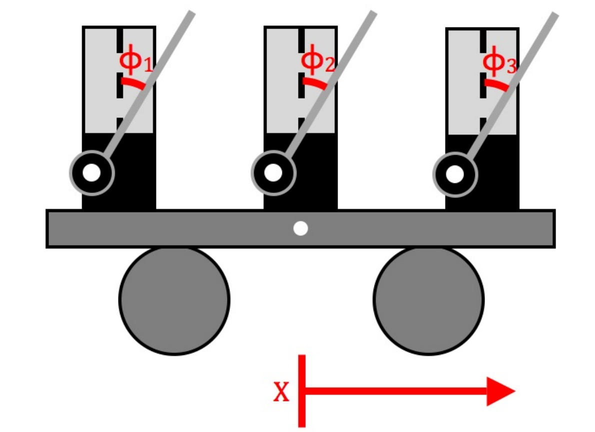

FIG. 1: Experimental setup for the case of $N = 3$. Variables are indicated. Note that this figure is not to scale.

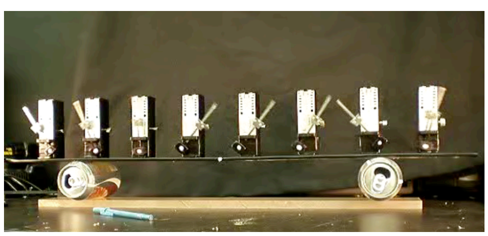

FIG. 2: A photo of the experimental setup for the case of $N = 8$.

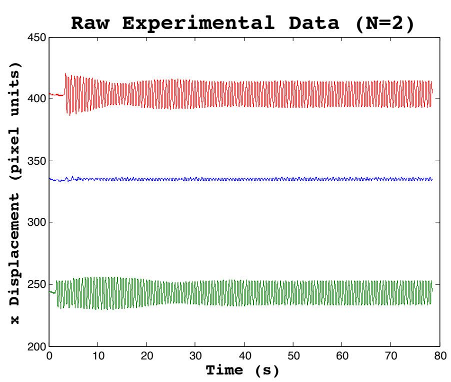

FIG. 3: Raw Labview data of the x displacement of 2 metronomes (red and green) and the platform (blue).

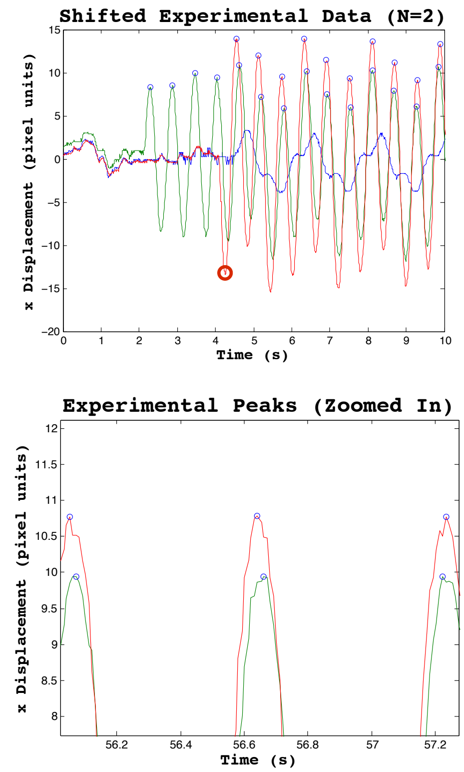

FIG. 4: (Top) A time series of x displacement where all oscillations have been shifted to the origin. The peaks found by Matlab's "findpeaks" function are indicated by blue circles. The peak which indicates the start of the Nth metronome and the release of the platform is indicated by a larger red circle. (Bottom) A zoomed in view of three experimental peaks. It is clear from this figure that the value returned from "findpeaks" is not necessarily the center of the particular oscillation.

III. THEORY

The equations of motion for two coupled pendulums are available in Bennett et al.[3] and Wiesenfeld and Borrero-Echeverry[6]. These equations may easily be extended to N metronomes:

$$\ddot{\phi}_j + b\dot{\phi}_j + \frac{g}{l}\sin\phi_j = -\frac{1}{l}\ddot{X}\cos\phi_j + F_j \quad (1)$$

$$(M+m)\ddot{X} + B\dot{X} = -ml\frac{d^2}{dt^2}(\sin\phi_1 + \sin\phi_2 + ... + \sin\phi_N) \quad (2)$$

where ϕ_j is the angular displacement of the jth pendulum, b is the pivot damping coefficient, g is the acceleration due to gravity, l is the pendulum length, X is the linear displacement of the platform, F is the impulsive drive, M is the platform mass, m is the metronome bob mass, B is the platform friction coefficient, and the dots represent differentiation with respect to time.

Nondimensionalizing results in the following:

$$\frac{d^2\phi_j}{d\tau^2} + 2\tilde{\gamma}\frac{d\phi_j}{d\tau} + \sin\phi_j = -\frac{d^2Y}{d\tau^2}\cos\phi_j + \tilde{F}_j \quad (3)$$

$$\frac{d^2Y}{d\tau^2} + 2\Gamma\frac{dY}{d\tau} = -\mu\frac{d^2}{d\tau^2}(\sin\phi_1 + ... + \sin\phi_N) \quad (4)$$

where we have introduced the dimensionless parameters:

$$\tau = t\sqrt{\frac{g}{l}} \quad (5)$$

$$\mu = \frac{m}{M+Nm} \quad (6)$$

$$\tilde{\gamma} = b\sqrt{\frac{l}{4g}} \quad (7)$$

$$\Gamma = \frac{B}{(M+Nm)}\sqrt{\frac{l}{4g}} \quad (8)$$

All parameters and variables described are well defined and can be experimentally measured or estimated, with the exception of $\tilde{F}_j$, the impulsive drive of the metronome, which receives a treatment first described by Bennett et al.[3]. The escapement mechanism which "kicks" the metronome is mimicked in two parts: first, the angular velocity of the pendulum is slowed by some factor γ, and then, a numerical constant c is added to the angular velocity:

$$\left|\frac{d\phi}{d\tau}\right| \rightarrow \gamma\left|\frac{d\phi}{d\tau}\right| + c \quad (9)$$

With a complete theoretical model, we must estimate the values of our parameters before we can fully construct a computer simulation of the experiment.

IV. PARAMETERS

Several parameters must be estimated for use in the simulation. With the exception of gravitational acceleration and the number of metronomes, all parameters must be estimated with a varying amount of uncertainty. The platform mass contains perhaps the least uncertainty, and is given (in kilograms) by $M = 0.0655 + N(0.094 - m)$, where we include the mass of the metronome minus the bob in the total. The pendulum bob mass is estimated as $0.022kg$; this is only estimated because an exact measurement would require the destruction of borrowed experimental materials.

The pivot damping coefficient, b, is estimated from a measurement of the decay time for an undriven oscillation of the metronome, experimentally determined as $t_{decay} = 20s$; the value used for b is thus $b = 2m/t_{decay} = 0.0022$. The platform damping coefficient, B, has a considerable amount of uncertainty; we expect that for our soda cans rolling on a smooth surface, the damping should be very small, so we somewhat arbitrarily pick $B = 0.001$.

As far as the parameters corresponding to the modeling of the escapement mechanism are concerned, γ is difficult to estimate, but c can be predicted fairly well. For c, this is because it is primarily the magnitude of this kick that determines the amplitude of the pendulum after several oscillations; we experimentally observe oscillations in the range of about 45°, so we can adjust the value of c accordingly. For our simulation, we have decided to use $\gamma = 0.97$ and $c = 0.025$, as was used in Wiesenfeld and Borrero-Echeverry[6].

The pendulum length l, the final remaining parameter, is difficult to estimate because of a simplification in the model, different from the metronome we use in our experiment. The simulation models a pendulum with a point mass, m, a distance l from the pivot. The metronome, in reality, has two spatially extended masses located at two different distances from the pivot. An attempt to calculate an equivalent distance resulted in $l = 0.025m$; however, this gives a natural frequency for the simulation that is considerably higher than what is experimentally observed. The simulation was run a second time with a new value of $l = 0.1m$ to match the natural frequency of the pendulums in the simulation to that of the experiment.

V. SIMULATION

Incorporation of equations (3) − (9) and the parameters listed above into a computer model allows for a simple numerical investigation for the cases of $N = 2$ through $N = 9$. For each case, we run 100 simulations with randomly selected initial conditions (similar to the "random" initial conditions of the experiment) of approximately constant total energy. The equations of motion are integrated in the simulation up to $\tau = 2000$, and synchronization is determined from two cri-

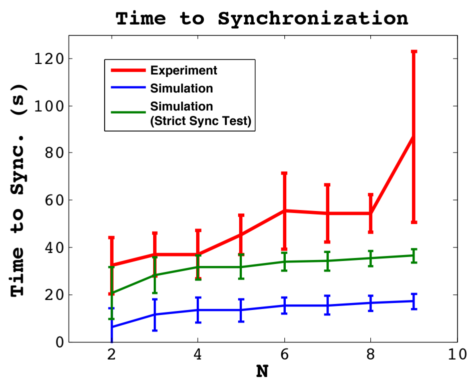

FIG. 5: The time to synchronization for experiment and simulation. Simulation parameters are our "best estimate," as indicated in Section IV above, with $l = 2.5cm$. Error bars indicate one standard deviation. The natural frequency of the simulation is about 3 times that of the experiment.

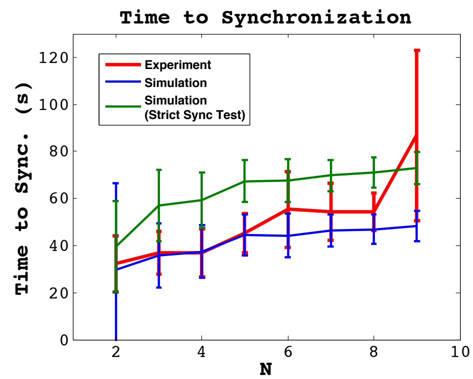

FIG. 6: The time to synchronization for experiment and simulation. Simulation parameters are our "best estimate," as indicated in section IV above, with $l = 10cm$. Error bars indicate one standard deviation. The natural frequencies of the simulation and experiment are approximately equal.

teria: (1) applying the same test from the experimental determination, but applied to the simulation ϕ_j variables, and (2) applying a "strict sync test" in which all peak ϕ_j values must be within one time discretization. Runs which fail to satisfy the criteria are noted.

VI. RESULTS

The time to synchronization for experiment and simulation is shown in Figures 5 and 6. Since the definition of synchronization is somewhat subjective, two synchronization tests are used for the simulation. The first, drawn in blue, uses the same criteria as for the experiment, applied to the time series of ϕ_j vaues. This line should be compared to the experimental result, but since the simulation is not subject to experimental error, this criteria may be too lenient. The second simulation line, drawn in green, uses a "Strict Sync Test" to determine synchronization; the condition for synchronization in this case is that all peak values of ϕ_j must be within one time discretization. These two lines are intended to provide a range of simulation synchronization times depending on the stringency of criteria.

Figure 7 shows the proportion of runs in both the experiment and simulation that satisfy the synchronization criteria within the allotted time. The decrease in synchroniza-

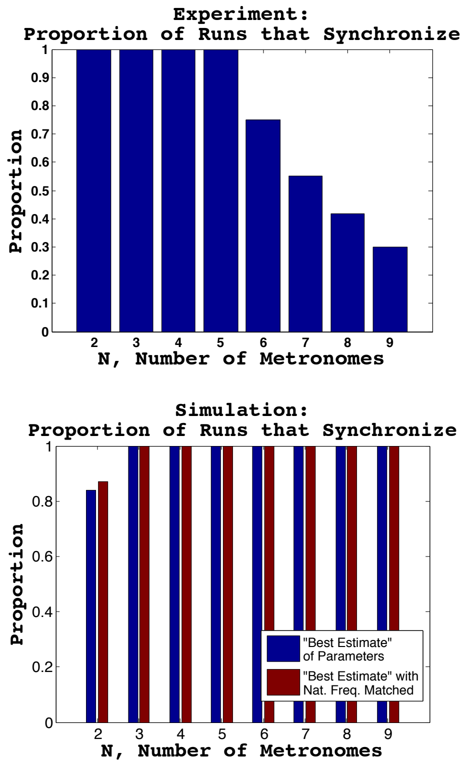

FIG. 7: (Top) The proportion of runs that synchronize within the allotted time.

tion for $N > 5$ in the experiment may be attributable to short data runs, typically of only 2 minutes. Had we waited longer, perhaps synchronization would have been reached; however, more than once we waited as long as the metronome winding would allow, and synchronization was not observed. For the simulation, synchronization is observed in every case except $N = 2$. Although this suggests that there are some fundamentally different dynamics between the experiment and simulation, this result in the simulation may be explained as follows. Wiesenfeld and Borrero-Echeverry demonstrated (see Figure 6 of Wiesenfeld and Borrero-Echeverry[6]) that varying the platform damping coefficient can cause the size of the basins of attraction for in phase and antiphase synchronization to expand or contract. We believe that for our particular set of parameters, which differ from those of Wiesenfeld and Borrero-Echeverry[6], a thin basin of attraction exists for the antiphase state; as we add more mass to the platform, we decrease the value of Γ and the antiphase state is no longer an attractor for $N = 3$ or higher.

VII. CONCLUSION

We extend recent theoretical developments in the modeling of coupled oscillators beyond the $N = 2$ case. We investigate the time to synchronization for N coupled metronomes both numerically and experimentally, and find considerable agreement between the two, especially when the simulation parameters are altered to match the natural frequency experimentally observed. Discrepancy between simulation and experiment may be attributed to a number of factors, including but not limited to: (1) incorrect estimates of parameters, (2) experimental complexities not captured by the model (e.g. the metronome "kick" mechanism, a reaction force from the platform, etc.), and (3) limited experimental and numerical statistics.

VIII. FUTURE WORK

Much work can be done to improve upon this effort. In particular, much work can still be done using the data we have collected. A better algorithm for determining synchronization (in place of our ϵ method) would likely give more reliable results from this data. Alternative methods include a weighted average to determine the peaks and perhaps a Fourier analysis. Further analysis of our data could extract results such as how frequency changes with N or a quantification of the non-synchronized states that we observed.

In terms of extensions beyond this work, obvious improvements include a better estimate of the simulation parameters and a larger ensemble of data for stronger statistics. Additionally, one could vary any of the parameters and test the agreement between experiment and numerics. The two most appealing parameters to vary are perhaps the natural frequency of the metronomes and the platform mass.

Clearly, there is much to be learned before the phenomenon of synchronization is thoroughly understood, but a solid foundation is being constructed.

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