The Astrojax Pendulum

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In this paper we investigate the properties of a generalized double pendulum. Previous investigations on double pendulums often limit it to a planar case, and fix the lengths of each stage of the pendulum. Astrojax is notable in that the pendulum stage lengths vary dynamically. We measure its behaviors in response to well quantified forcing using a perpendicular camera array to track position, and from that generate three-dimensional trajectories for each pendulum bob. The trajectories' mean chaotic lifetime and dominant oscillation frequencies are analyzed, and we find that this chaotic pendulum's oscillation frequencies are linearly correlated with the forcing frequency in the low-frequency regime. We also conclude that a naive periodic forcing of this system without a feedback mechanism is not sufficient to create stable, long-lived orbits of the Astrojax pendulum.

I. INTRODUCTION

A. General Framing

Oscillatory mechanical systems are a subset of dynamical systems, which can describe the evolution of states for nearly all physical phenomena. Oscillations of various types occur in real world mechanical systems, nearly all of which are thermodynamically irreversible (via damping, friction, loss of energy to heat, etc.). In this study we will specifically study a set of coupled, chaotic oscillations formed from a forced double pendulum.

Pendulums are one of the most fundamental physical systems studied, and while they may be common, they are only ideal in the most simple cases. While a single pendulum can have its equations of motions solved simply, especially in the low amplitude limit, a "double" pendulum has coupling that gives rise to chaotic behavior.

The traditional double pendulum has rigid axes, which defines the distance that the pendulum's two bobs can be from each other and from the pivot. It is also constrainted to a two-dimensional plane, having a fixed polar angle. This configuration in total gives the traditional double pendulum two degrees of

FIG. 1. A rigid-axis double pendulum

freedom: θ_1 and θ_2, the two azimuthal angles of the pendulum axes from their respective pivots. It has been shown that such a double pendulum demonstrates chaotic behavior, but we want to examine the kinds of behavior that occur when we increase the degrees of freedom from two degrees to **five** degrees. Such a pendulum can be easily found in a simple toy: the Astrojax.

B. Specific Introduction

The Astrojax is essentially an assembly of three weighted spheres on a string, with the middle ball having freedom to move along the string, hindered by some small friction. The Astrojax is a more general double pendulum, with constraints removed in that the central bob is free to move about the string, and that the polar angles are not constrained, allowing it to move in a 3D space. This gives the Astrojax **five** degrees of freedom:

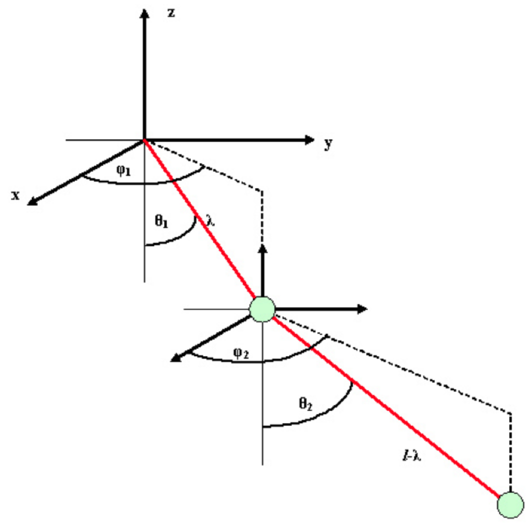

FIG. 2. Diagram of the Astrojax Pendulum, including the five degrees of freedom

1. λ – The distance between the first and middle pendulum bob
2. θ_1 – The first axis' azimuthal angle
3. θ_2 – The second axis' azimuthal angle
4. ϕ_1 – The first axis' polar angle
5. ϕ_2 – The second axis' polar angle

These additional degrees of freedom will hopefully give rise to even more interesting chaotic behavior; if a rigid double pendulum already has chaotic properties, an unconstrained one is presumed to be even more chaotic, especially under specific forcing conditions. Some side benefits to consider in studying this system include:

- The system has chaotic properties, but it is simple for a non-scientist to pick up and handle, unlike something like a condensed matter or plasma system that exhibits chaotic oscillations.

- The Astrojax is a consumer level toy, easily purchased for 15 dollars or less for any layperson.

- The Astrojax exhibits interesting behavior with non-precise forcing momvements; a human can easily create chaotic oscillations using only their hand as a forcing tool.

II. METHODS

A. Description of Experimental Setup

The motion of the Astrojax is, by nature, extremely complex. The motion of a double pendulum constrained to move in only two dimensions is already chaotic, and the sheer complexity of allowing motion in three dimensions can easily be extrapolated. It also bears stating that Astrojax is not purely a double pendulum. The center mass is free to move along the string, allowing the length of the two coupled pendulums to vary, but it is, fortunately, constrained to stay on the string itself, between the two end masses.

In order to proprerly observe, quantify, and analyze the data we made use of the Optitrack motion capture system and its proprietary software MOTIVE. A DENSO robotic arm held one of the end masses of the Astrojax and forced it, resulting in oscillations.

We forced the Astrojax in the vertical, or z direction in a trianglar wave rhythm using various speeds and amplitudes of the robotic arm. The software WINCAPS III was used in order to interface and program the robotic arm. The Astrojax were covered in infrared reflective tape for purposes of motion capture, along with a point on the robot arm. Thus, we could capture the three-dimensional dynamics of the two free bobs along with the forcing trajectories. Once captured and trajectorized, Optitrack's interpretation of marker disappearances and swaps in vertical position resulted in data gaps, marker identity loss, and marker switching. A tracking program in MATLAB was implemented (credit to M. Kingsbury) which used a ternary search algorithm which searched for least position differences in order to resolve each marker into a continuous path. Therefore we could achieve full 3-D trajectories for the middle & end bobs, along with the forcing marker.

In addition, we investigated the forcing motions used by a human in creating stable orbits with the Astrojax. To do this, we attached an infrared reflective marker to the human forcer's hand via a glove, and implemented the same methodology as with the robotic forcing, simply replacing the forcing marker.

B. Physical Interpretation of the System

Previous analysis of the Lagrangian[1] and the Newtonian[2] equations of motion have been derived in previous works. We provide a summary of these equations below. The generalized cooridnates are given as

$$q = [x_1, y_1, z_1, x_2, y_2, z_2]$$

This allows us to write the Langragian of the system as

$$L = \frac{1}{2}\dot{q}^T M \dot{q} - V(q),$$

where

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix}$$

and

$$V(q) = g(m_1 z_1 + m_2 z_2)$$

The Newtonian form is

$$a_{2x} = \frac{Fs_{1x} - Fs_{2x} + Fd_{2x}}{m_2}$$
$$a_{2y} = \frac{Fs_{1y} - Fs_{2y} + Fd_{2y}}{m_2}$$
$$a_{2z} = \frac{Fs_{1z} - Fs_{2z} + Fd_{2z}}{m_2} - g$$
$$a_{3x} = \frac{Fs_{2x} + Fd_{3x}}{m_3}$$
$$a_{3y} = \frac{Fs_{2y} + Fd_{3y}}{m_3}$$
$$a_{3z} = \frac{Fs_{2z} + Fd_{3z}}{m_3} - g$$

where a_{ij} is the acceleration of the i^{th} mass in j direction, g is the acceleration due to gravity $Fs_{\alpha\beta}$ is the force of the string on mass α in the β direction, and $Fd_{\alpha\beta}$ is the drag force and follows the same conventions as above.

While these equations provide a good interpretation of the Astrojax pendulum with a taut string, we find they do not cover a great deal of cases that occur in the real system we tested. For example, the Newtonian equations do not cover cases where the string goes slack unless the tensile force component is made piecewise active. This piecewise behavior can make the set of ordinary differential equations analytically intractable for traditional analysis methods. In addition, the real system where Astrojax bobs with a finite radius are physically touching is not accounted for, as the actual degenerate state of the two bobs touching does not have them located at the same position.

III. RESULTS

We recorded an average of three takes each for each amplitude of robotic forcing, and five takes each for each type of human forcing (Horizontal, Vertical, Butterfly). In general, the naive method of having the robot arm force the Astrojax with a periodic motion with no feedback did not result in any stable orbits occurring, only quasi-stable and/or chaotic orbits that rapidly change or decay. In the robotic forcing, we found that a sufficient degree of acceleration was required in order to separate the two bobs from one another and cause the system to evolve away from the trivial state (i.e. the two bobs remain in contact with one another at the string's nadir). Once separated in the robotic forcing, we found the Astrojax did not follow any predictable behaviors; their motion was mostly chaotic under this simple periodic forcing. The duration of time where the bobs were in this chaotic behavior vs. the time they spent in the trivial state increased overall as the forcing progressed, as shown in the plots below. We also plot the mean active time of each take vs. each forcing's respective amplitude and frequency. As seen in Fig. 6, there is high variance in the mean active time solely for a forcing amplitude of 150mm, while the remaining amplitudes cluster around having 30 seconds. Figure 7 shows the same data with respect with the forcing frequency. Note that a small difference in forcing frequency near 2.5 Hertz seemed to generate a wider spread of mean active times.

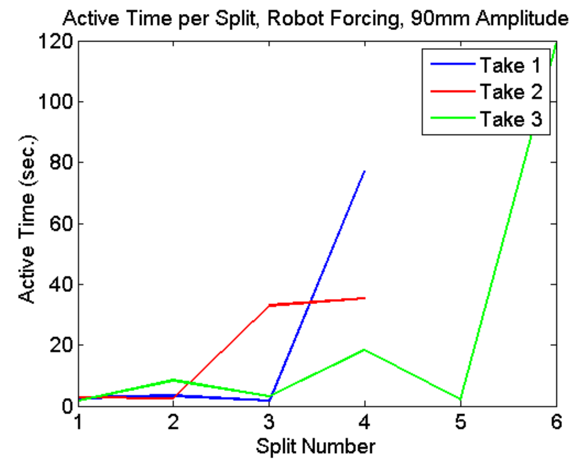

FIG. 3. Plot demonstrating the active time vs. the Number of Splits for each take of the forcing Amplitude of 90mm.

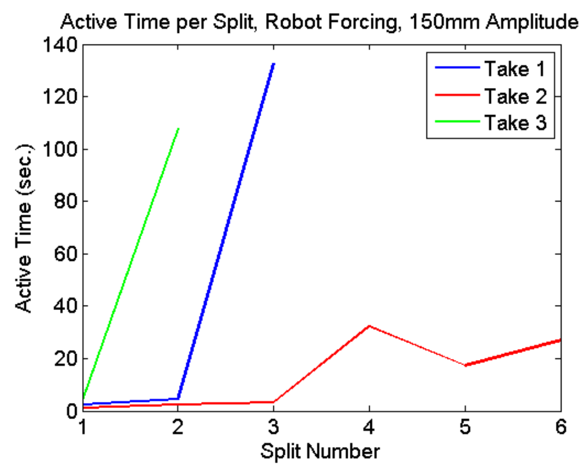

FIG. 4. Plot demonstrating the active time vs. the Number of Splits for each take of the forcing Amplitude of 150mm.

Using MATLAB's Fast Fourier Transform (fft), we found the most dominant frequency in the recorded oscillations in all three Cartesian dimensions. The resultant frequencies were often symmetrical about the Cartesian x & y coordinates, while ffts of the z coordi-

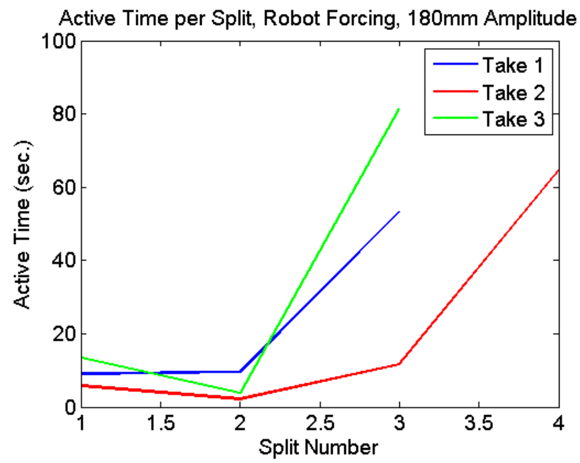

FIG. 5. Plot demonstrating the active time vs. the Number of Splits for each take of the forcing Amplitude of 180mm.

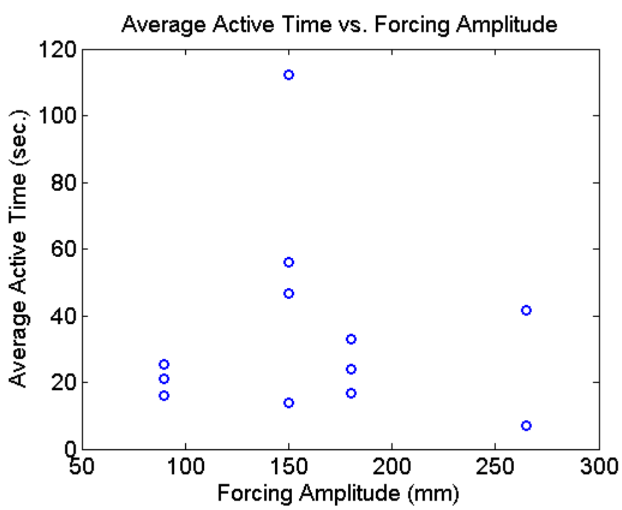

FIG. 6. Plot showing Active Time. vs. the Forcing Amplitude for Robotic Forcing

nate often failed or were too noisy for some unknown reason. The plot below shows the resultant frequencies of each bob as a function of each take's forcing frequency.

As seen in Fig. 8, as the forcing frequency increases, the resultant dominant oscillation frequency for each bob tends to increase in a

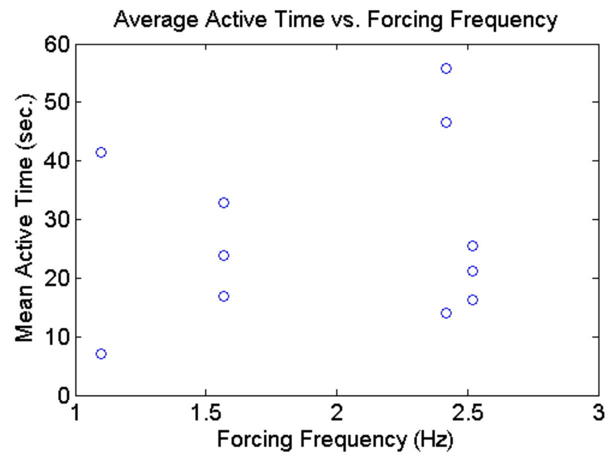

FIG. 7. Plot showing Active Time. vs. the Forcing Frequency for Robotic Forcing

Linear Regression ($y = ax + b$)			
Jax	a	b	R^2
Mid	0.145	0.225	0.719
End	0.242	0.0065	0.706

TABLE I. Linear Regression Parameters for Best Fit Lines in Fig. 8

roughly linear manner, with the end bob increasing slightly faster than the middle bob.

With M. Kingsbury's tracking program, we created full 3-D plots and movies of the tracked trajectories. We provide one of the more interesting plots below: a trajectory plot of human forcing creating a stable horizontal orbit (Fig. 9).
The interesting part of the forcing is the green trajectory created by the human forcing marker. As seen in the plot, a complicated, helical path in order to maintain a

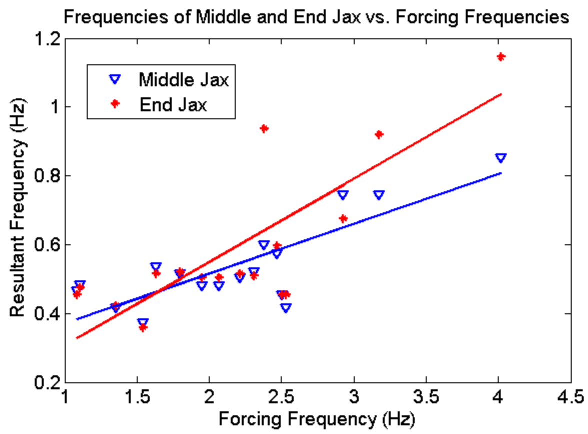

FIG. 8. The resultant mean dominant frequencies of the free bobs have an approximately linear correlation to the forcing frequency.

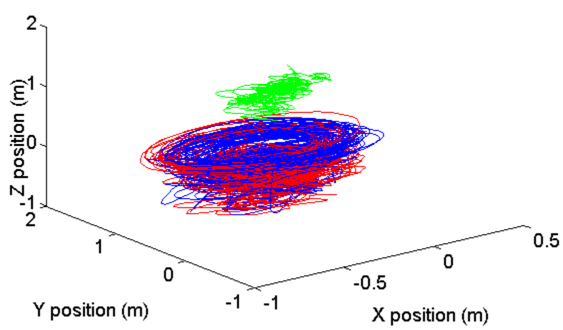

FIG. 9. 3-D Trajectory map of a Human-Forced Horizontal orbit

steady horizontal orbit. The dynamic path that the human forcing creates with changing frequencies and amplitudes implies that some human 'feedback' mechanism is necessary to create such stable orbits.

IV. DISCUSSION

From our results, we can qualitatively confirm certain aspects of the Astrojax system. As seen in Figure 8, the resultant frequencies of the middle & end bobs increase linearly as the forcing frequency increases. We suspect that at higher forcing frequencies this linear relation will no longer be true as the forcing frequency becomes too fast and the end bobs no longer have time to fall sufficient distances to split. Figures 3 thru 5 demonstrate that the active time of the end bobs increased as the Astrojax split from each other. These results match our qualitative observations with the robotic forcing. Often what would happen is that the forcing would generate small, short-lived splits of the jax that would either rapidly decay or create small, unstable orbits. It was only after some time of forcing and a few of these small splits that the jax would separate enough to create the full, rich, chaotic orbits with active times in the dozens of seconds.

Another observation we made in our experiments was that under certain smooth motions of the robotic arm's end affector, the Astrojax would not deign to split under any attempted forcing frequency or amplitude. This occurred because the type of motion interpolation used was too smooth, and the robot arm was slowing down before reaching its assigned endpoint. Using a different kind of interpolation generated a much jerkier motion and successful splitting of the Astrojax. Thus, we strongly suspect that a sufficient degree of acceleration is required to separate the bobs, an acceleration which most likely must exceed g. This acceleration is generated by a sufficiently jerky motion, since acceleration is the antiderivative of jerk.

V. CONCLUSION

The Astrojax is a very complex system with a great deal of dynamics and oscillatory patterns, especially when real-world constraints and perturbations are accounted for. From our results, we found that it is not feasible to generate stable orbits of any kind using a simple, naive periodic forcing. The best we could do was to create chaotic orbits that had some mean lifetime before eventually decaying. When examined alongside the complicated motions of human forcing that were necessary to create stable orbits, we conclude that some kind of feedback mechanism is needed to adjust the forcing frequency and/or amplitude on the fly to maintain a stable orbit.

Future work on the Astrojax system could involve creating a 3-D ordinary differential equation solver to simulate the trajectories of the Astrojax under our attempted forcings. We could then compare experimental vs. simulated trajectories and find differences in resultant frequencies and mean active times that arise from real-world constraints. Since we still have the raw data of the forcing trajectories and the initial conditions of the system, it would be simple to input those parameters into a ODE solver like MATLAB's Simulink library and examine differences in resultant trajectories that arise from the nonlinearities and chaos inherent in this system.

REFERENCES

[1] P. Du Toit, "The astrojax pendulum ad the n-body problem on the sphere: A study in reduction, variational integration, and pattern evocation,".

[2] D. Dichter and K. Maschan, "Modeling and simulation of astrojax,".