The Effects of Connection Topology on Synchronization Time in a Population of Fireflies

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(Dated: 14 December 2012)

Examples of spontaneous synchronization abound in the biological world, with examples ranging from the oscillations of cardiac pacemaker cells to the synchronization of firing neurons. A particularly spectacular example is the emergence of synchrony that is observed when groups of fireflies of a certain species congregate and flash. In this experiment, we use an experimental model consisting of 48 flashing LED lights, monitored by a camera, to study the dynamics of fireflies interacting according to a modified Kuramoto model. The strength of coupling and number of connections are both varied over multiple trials, during which the system is allowed to iterate a finite number of times, at which point the variance in oscillator phase is recorded. From our collected data, we find evidence of a power law relation between the time until synchronization and the number of connections between fireflies in the system.

I. INTRODUCTION

The flashing of fireflies is a familiar biological phenomenon that has been studied for hundreds of years[1]. Its purpose is currently understood as a courtship behavior used during the mating season of the particular species, with usually mobile males flashing to signal typically stationary flashing females[1]. As is the case for many biological oscillators, it has been observed[2] that when a population of fireflies flash in response to each other, coupling their dynamics, the characteristically nonlinear phenomenon of synchrony can occur, with the fireflies all flashing with identical frequencies. The exact biological and evolutionary advantages of this behavior are still a topic of debate among scientists, with several models proposed at the time of writing of this paper. Just as controversial is the question of which dynamical system best describes this behavior, with paths to synchronization proposed by many scientists, such as Avila[3] and Ermentraut[1], and a general model for pulse coupled biological oscillators from Strogatz, which he has formally proven to produce synchrony for the case of N oscillators, $\forall N \in \mathbb{Z}^{(+)}$[4].

In this formulation, the oscillators are globally coupled and dynamically identical. Each firefly is assigned a state variable, x, such that $x = f(\phi)$, where $f : [0,1] \to [0,1]$ is smooth, increasing monotonically, and concave down. When the state x_i of an oscillator reaches the threshold at 1, the state of each oscillator x_j in the system is given a perturbation ϵ defined such that $x_j(t^+) = \min(1, x_j(t) + \epsilon)$[4] (Fig. 1). This model, while less biologically interesting than others, provides motivation for the "integrate and fire" style of dynamics that we chose to model.

With such a wealth of biological and mathematical background, our first goal was to determine which model to use in our experiment. We chose an alternative to the Strogatz model, the mean-field variant of the Kuramoto model (1).

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad (1)$$

Where θ is an angular state variable for the indexed oscillator, defined on the interval $[0, 2\pi]$, ω_i is the natural frequency of the oscillator, K is the coupling strength, and N is the number of other oscillators in the ensemble to which the i^{th} oscillator is coupled. For a more formal bifurcation analysis of this system, see our wiki page - the full calculation and proof has been omitted from this paper in the interest of space. For our experiments, we used our own modified variant of this equation, in which the summation was carried out only over the subset of oscillators that had recently fired. The system for which the model was implemented is a 6×8 grid of fireflies (oscillators) physically modeled as a 6×8 grid of uniformly spaced LEDs.

II. METHODS

A. Model

The mathematical model we used for our experiment draws its inspiration primarily from the previously described Kuramoto model. Unlike the original Kuramoto dynamical system, when calculating the state of the i^{th} oscillator, our model sums only over the subset of oscillators that are coupled to the i^{th} oscillator and have fired at the given time.

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j \in fired}^{N} A_{ij}\sin(\theta_j - \theta_i) \qquad (2)$$

Where A_{ij} is a symmetric 48×48 matrix such that $A_{ij} = 1$ if the j^{th} oscillator is coupled to the i^{th} oscillator, or $A_{ij} = 0$ if the i^{th} os-

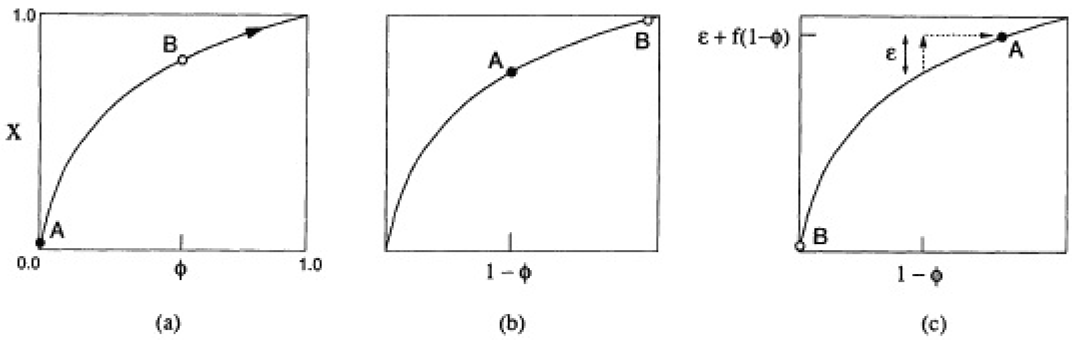

FIG. 1. An illustration of the dynamics of two oscillators in Strogatz' model[4]

cillator is not coupled to the j^{th} oscillator. The matrices were generated algorithmically in MATLAB, and were distinguished by radius and number of connections allowed per firefly. Each radius choice corresponds to a unique number of connections, and therefore the two numbers are interchangeable in describing a given coupling matrix. The matrix allows us to treat the number of other oscillators coupled to a given oscillator as an independent variable, by ensuring that the dynamics of oscillator i will only be affected by the flashing of oscillator j if the two are coupled. This dynamic is biologically motivated by the fact that a given firefly in a forest will have a finite range of vision, and therefore will only respond to other fireflies within a certain radius. (Fig. 2) For modeling purposes and to legitimize the number of connections as a variable, the system is treated as isotropic, such that one may imagine the grid being repeated at each edge (for example, if you were to walk off of the left edge, you would find yourself facing the oscillators on the right side).

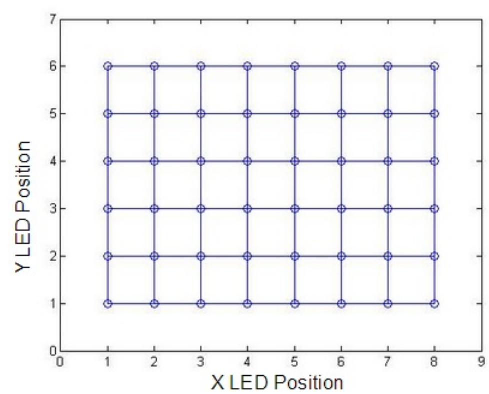

FIG. 2. A visualization for the connections for r=1. Each point represents a firefly, and any two fireflies connected by a blue line are coupled.

Implementation of this model computationally is then a simple matter of, for each oscillator i in the ensemble, iteratively summing over the phase contributions of each other oscillator that has fired at each timestep, dt, and updating the phase of the oscillator i accordingly. The matrix A_{ij} will take care of any non-coupled oscillators in the sum by setting their contribution to zero.

B. Experiment

The experimental setup was a 6×8 grid of LED lights soldered onto a pre-manufactured LED board. The LEDs were controlled by a micro-controller based on the Arduino platform, which in turn was programmed using MATLAB. Above the board, a camera was positioned such that all 48 LEDs were within its range of sight. The camera was programmed in MATLAB to detect, at each time step, which LEDs were flashing. This visual input was then processed by MATLAB, which then communicated with the LED array through the micro-controller, telling which LEDs to flash in the next time-step.

The coupling matrices, A_{ij}, we tested were those associated to radii of 1, 2, $\sqrt{5}$, 3, $\sqrt{13}$, 4, $\sqrt{20}$, 5, and 6, which correspond to 4, 8, 12, 20, 27, 33, 40, 46, and 47 connections per firefly, respectively. The numerical values of K examined were 1, 2, 5, 7, 10, 15, 20, 25, and 30. For each coupling matrix, the experiment was iterated for each K value, and run for 15 trials with randomized initial conditions chosen such that $\theta_{i,initial} \in [0, 2\pi], \forall i \in [1, 48]$. The relationship between visual radius and number of connections is shown explicitly in Fig. 4. For the firefly in the bottom left corner, each circle of a given color corresponds to a radius. The fireflies on or within the circle are coupled to this firefly, with the visual radius being projected across the grid for each coordinate such that the visual radius of each firefly is isotropic. In order to ensure that our experiments were run without interference from outside light sources, we placed a cardboard box over the board and camera, as can be seen in Fig. 3. When an experiment was run, the MATLAB code that operated the camera and LEDs was allowed to run for a set number of loop iterations. At the end of the cycle, the mean squared error in the phases of the oscillators was computed, using the formula $E = \frac{1}{N}\sum_{i=1}^{N}(\hat{X}_i - X_i)^2$, where $\hat{X}_i$ is a vector containing the predicted values (all phases equal), and X_i is a vector containing the true values (the state of the system at the iteration at which the error is calculated). Initially, the cycle was run for 3000 loop iterations, but this was found to be insufficient to see synchronization in smaller radii, so this number was increased to 6000 for later trials. The system was considered to be synchronized when the error of the oscillator phase dropped below a threshold of 1×10^{-30}.

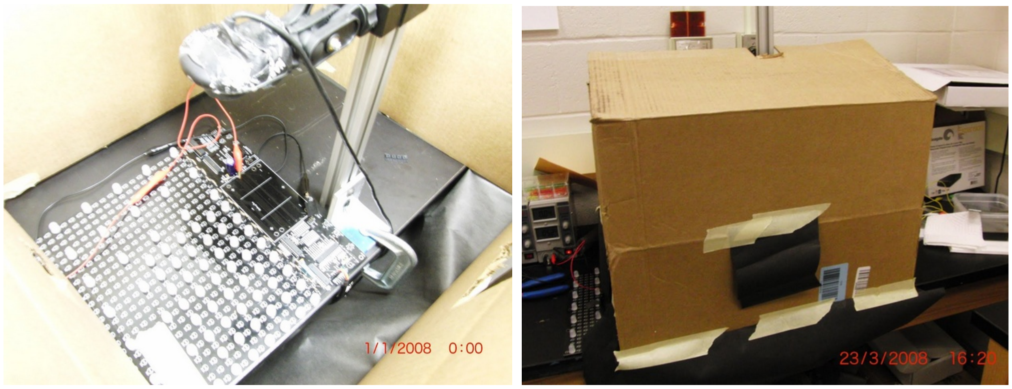

FIG. 3. Left: The LED board, with camera overhead. Right: The box used to prevent interference from other light sources in the lab.

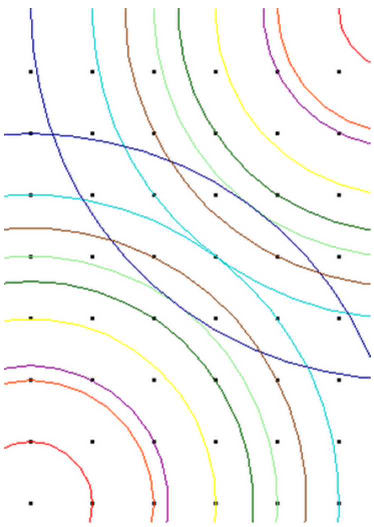

FIG. 4. A visualization of how radius corresponds to connections.

III. RESULTS

For each pair of coupling strength and connection matrix, the data for number of oscillators fired per cycle (Fig. 4) and number of iterations until synchronization (Fig. 5) was recorded (in all plots, the number after the word error corresponds to the radius of connections). The relationship between coupling strength and time until synchronization for several coupling radii is summarized in Fig. 7.

From these plots, it is apparent that the time until synchronization decreases as the coupling strength increases. This is to be expected, as a higher coupling strength implies that a given oscillator's phase is more strongly affected by its neighbor's phase. The full relationship between time until synchronization and coupling strength can be seen from Fig. 8, where the curves are exponential best fit curves for different connection numbers. The relationship between number of connections and synchronization time, plotted for different values of K, is represented in Fig. 9. It is apparent from this data that larger radii, and therefore increased number of connections, leads to shorter synchronization time. In fact, it is possible to represent this with an exponential relationship between

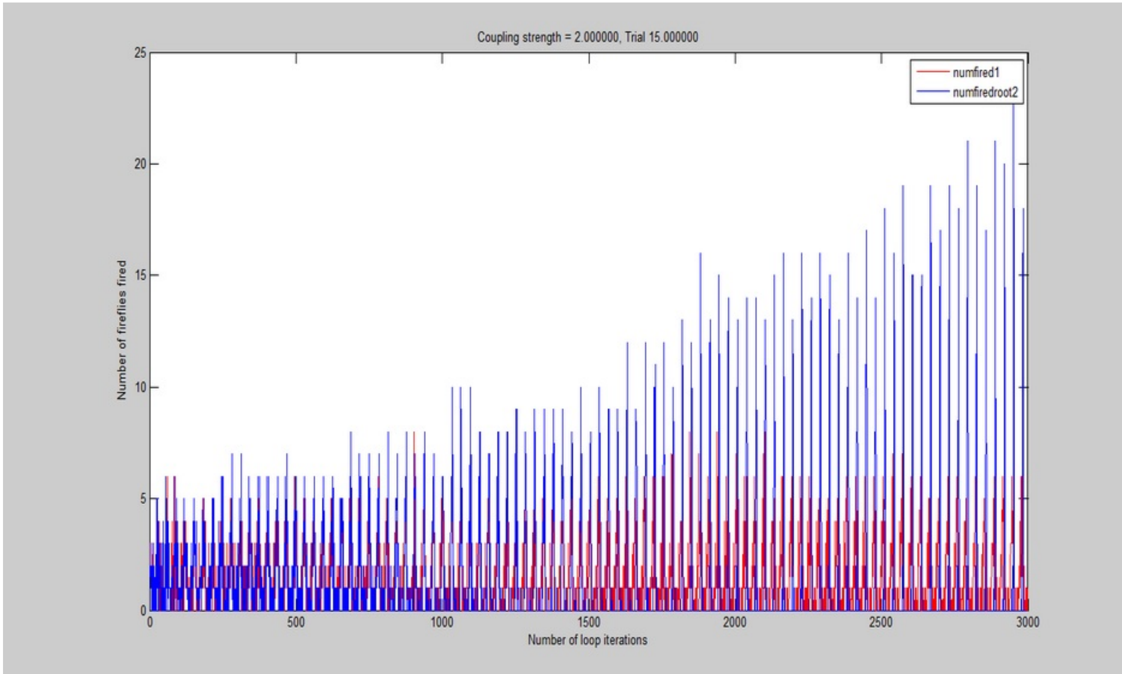

FIG. 5. Number of oscillators firing at each iteration, for K=2, at trial 15.

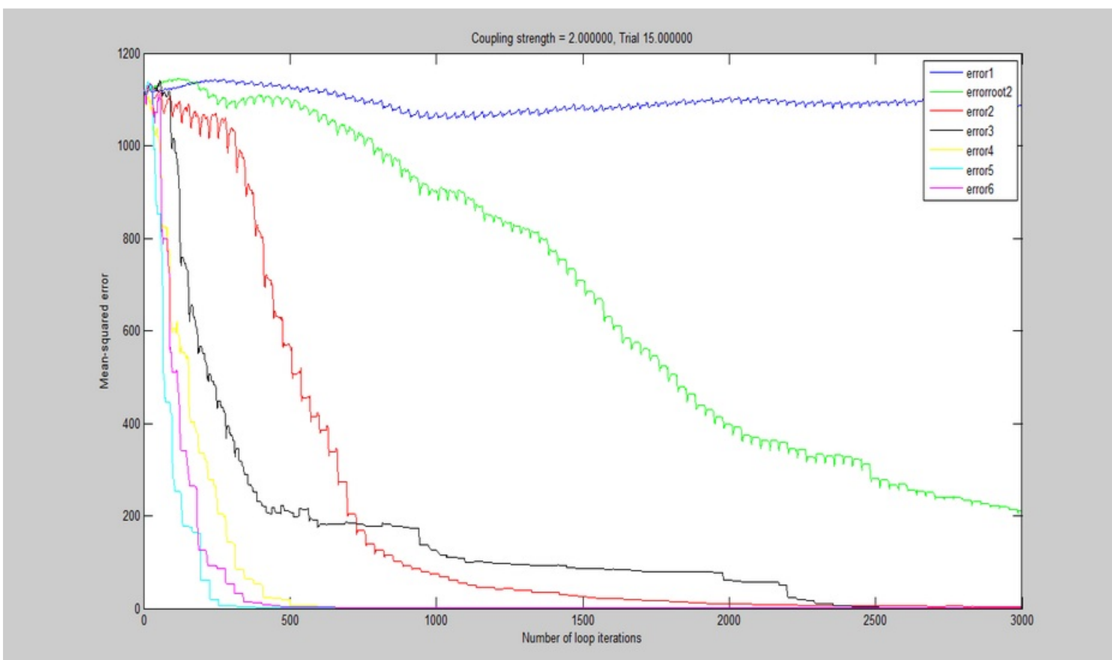

FIG. 6. Mean squared error of phases at each iteration, for K=2, at trial 15.

the time until synchronization, which we call τ, the coupling constant, K, and the number of connections, c. The exponential curve coefficients, a and b, for connection number are plotted in Fig. 10. Plugging these numbers into the general form for an exponential relation yields:

$$\tau = a * e^{(bK)} \Rightarrow \tau = \frac{18800}{c^{0.67}} e^{\left(\frac{K*(1.3c+42.5)}{1000}\right.} \quad (3)$$

IV. DISCUSSION

There were a few possible sources of error during our experiment. Initial data collected during the experiment had to be thrown out, as a deficiency was found in the algorithm for generating our coupling matrices. All of the data presented here was collected using the correct coupling matrices. However, because of time constraints, more data collection for larger iteration numbers would be needed to more accurately determine the relationship between the synchronization time and the independent variables. In particular, some of the connection numbers associated with smaller radii will definitely need longer times to synchronize in order to properly fit them on an exponential curve. Second, because experiments ran for long periods, it is possible that there was some optical noise detected by the camera, as the box used to shield the board had holes and creases. Further experiments should also be run for a wider range of N and K values in order to explore the possibility of bifurcations at certain K to N ratios. Many improvements could be made to the model, such as inclusions of the biological factors discussed by Ermentrout[1] and Avila[3]. These include the addition of phase shifts for finite response times, and the modeling of dynamics that are not based on a pure integrate and fire dynamic, as the Kuramoto model is. Interesting behavior that could be studied includes the formation of Chimera states[5] in non-locally coupled networks, and the pattern formation that can be observed in models with finite response times[6].

V. CONCLUSION

A great deal was learned while working on this project, if not about the dynamics of fireflies, then certainly about experimental and theoretical methods that may be used in further studies. The hint of a mathematical relationship between the time until synchronization for a population of fireflies, the number of fireflies which they can see, and the strength of coupling between any two fireflies is certainly present, and is approximated here as an exponential law. The qualitative dynamics of the system are clear from the data, with larger values for K and greater coupling radii/increased connection number corresponding directly to faster synchronization.

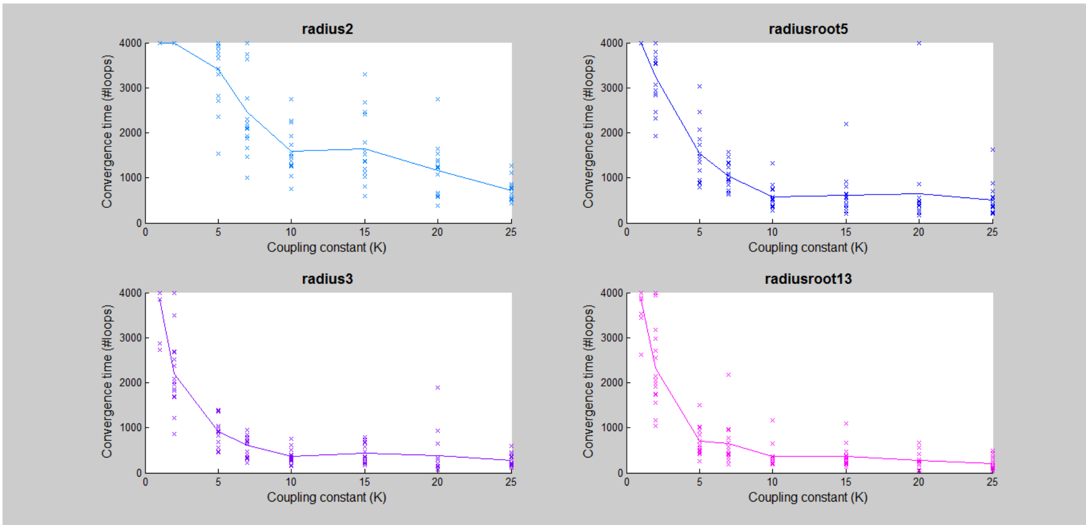

FIG. 7. A plot of coupling strength versus time until synchronization for various coupling matrices.

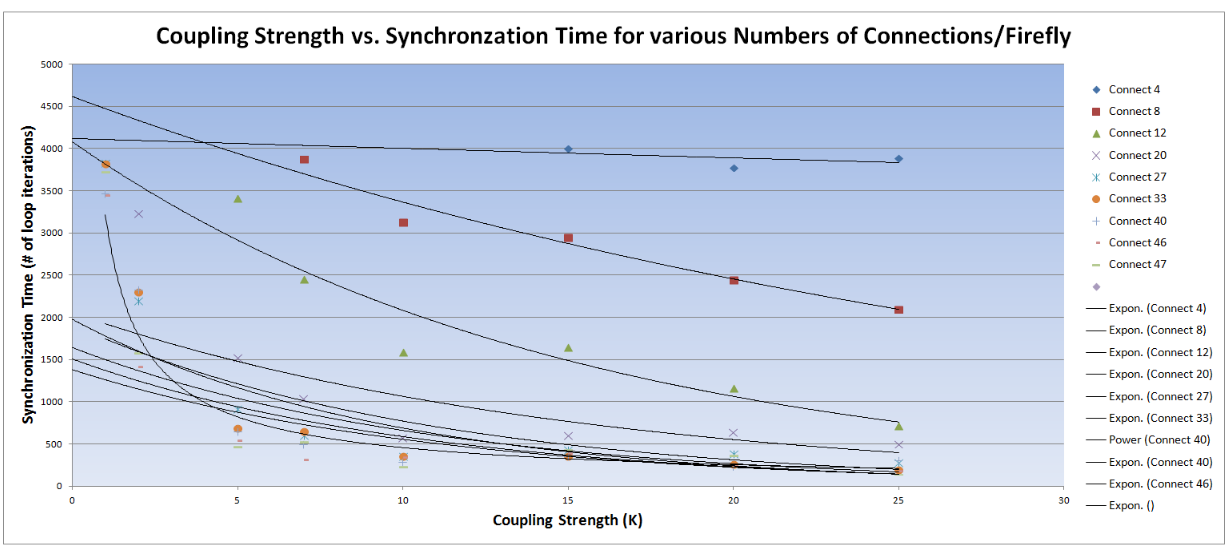

FIG. 8. Mean synchronization time for different values of K.

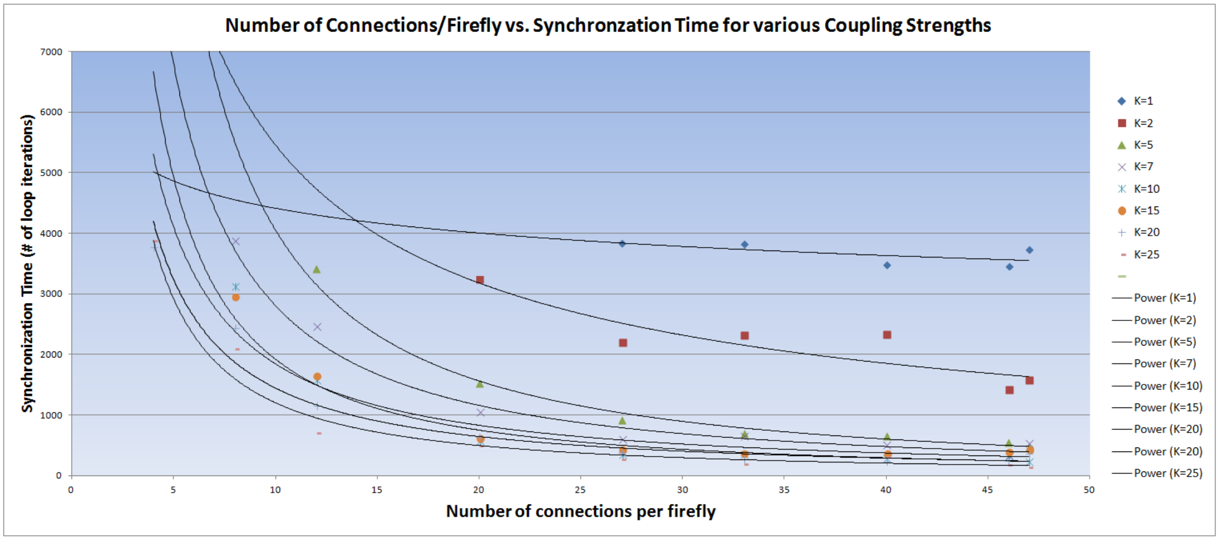

FIG. 9. Mean synchronization time for different connection numbers (radii).

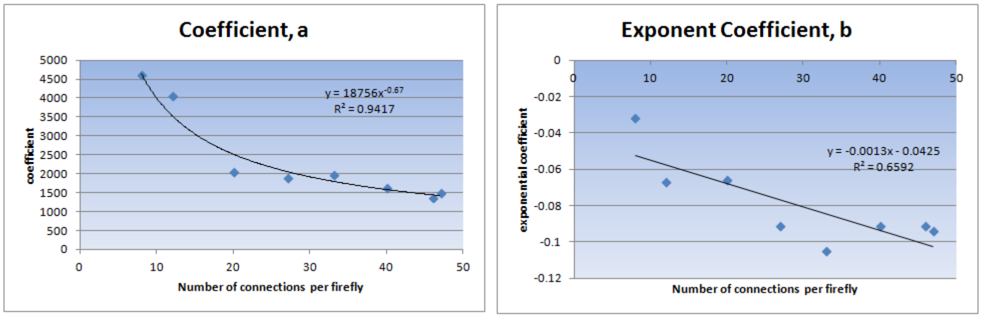

FIG. 10. Approximations for a and b coefficients in equation for τ.

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