An Investigation of the Chaotic Dripping Faucet

Nick Pritchard[1]

School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

(Dated: 16 December 2011)

The dripping faucet experiment has been performed multiple times since it was proposed by Robert Shaw in 1984 as a system exhibiting chaos. It continues to serve as a relatively simple chaotic system for numerical and experimental analysis. This project seeked to replicate past experiments and find the route to chaos in droplet formation. It was desired to collect enough data to construct a bifurcation diagram in order to predict system behavior at various flow rates. Additionally, simulations were performed by modeling the water droplet as a harmonic oscillator. Although there were no novel findings in this project, there were many significant experimental improvements that can aid others trying to perform the experiment.

I. INTRODUCTION

The dripping faucet experiment seeks to provide a relatively simple visualization of a system that exhibits seemingly predictable dynamics that quickly transitions to an entirely unpredictable, chaotic behavior under certain conditions. Through a straightforward experimental setup, the time between drips of a faucet is measured using a photodiode and a laser. The number of drips are dictated through precise control of the faucet head of reservoir control. This could be achieved by changing the flow rate or altering the water level of the reservoir tank that supplies the water. The photodiode records this drop as the water drop temporarily distorts the laser path to the diode. The photodiode is interfaced to a computer through an analog to digital converter. The data is easily analyzed and visualized through the computer.

The expected results show two main regimes: a period doubling regime, and a chaotic regime. The system gradually transitions from a single period to double period, followed by four periods. Once the flow rate is increased past these period doubling regimes, the time between drips becomes apparently unpredictable. Plotting the current period against the next period, the outwardly chaotic behavior begins to show signs of an attraction basin. This represents a classic example of chaotic behavior of a system that seems to contain easily understandable system dynamics and physical phenomena. Additionally, the seemingly random behavior after the initial period doubling also shows the basins of attractions that seem to exist for the chaotic regime.

II. BACKGROUND

The simplest model for the chaotic dripping faucet is that of a mass-spring system described in "The Chaotic Behavior of the Leaky Faucet"[1]. This system uses a simple differential equation of a mass-spring system whose mass increases linearly with time. When the position of the center of mass of the drop reaches a critical value, the mass of the drop is reduced proportionately to the velocity of the drop, simulating the pinching off of the droplet. Physically, this model is an intuitive description of the system: the water droplet is a mass, and is increasing linearly with the flow of water into it, which has some restoring force due to surface tension and some drag due to viscosity. The equation of motion for this model is:

$$\frac{d(mv)}{dt} = mg - ky - bv$$

where $d(mv)/dt = Q$ is flow rate, g is the acceleration due to gravity, k is the spring constant (coming from surface tension), b is the drag force (coming from viscosity), and $v = dy/dt$.

This model has been the most popular for the chaotic dripping faucet since its introduction. However, it is not very precise, and gives very little information about what are the appropriate values of the parameters (k, b, etc). This renders it a good qualitative model (exhibiting period doubling and transitions into chaos for some regions in the parameter space), and simple pedagogically, but not particularly accurate in an experimental context.

To rectify this, some[23] have employed far more detailed hydrodynamical models of the flow, using simplified versions of the Navier-Stokes equations. These models are far more accurate, but due to their added complexity, they are of limited use for gaining an intuitive understanding of the dynamics. Kiyono and Fuchikami[2] used numerical simulations based on one such hydrodynamical model to produce a modified version of the mass-spring system, this time including well-defined parameters gleaned from the simulations. Their equation of motion is:

$$m\ddot{z} + (\dot{z} - v_o)\dot{m} = -kz - y\dot{z} + mg$$

where z is the position of the center of mass, v_o is the velocity of the flow, $dm/dt = \pi a^2 v_o$ where a is the radius of the aperture, y is the drag force, and g is acceleration due to gravity.

For Kiyono and Fuchikami, this model

provided good agreement with their more exact hydrodynamical simulations, and exhibited period doubling and the transition to chaos in agreement with experiment. The mass of the droplet at each pinch-off event uniquely determines the mass of the next droplet, making this model a one-to-one mapping (the simplest that can exhibit chaos). Their model also shows hysteresis, and multiple regions where there is a transition to chaos.

III. SIMULATION

The dripping faucet system was modeled using the mass-spring model given in Kiyono and Fuchikami[2] with a 4th-order Runge-Kutta algorithm that was written by the authors.

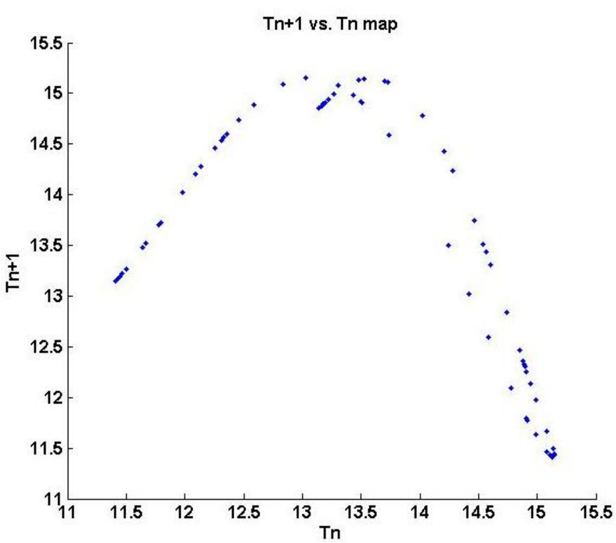

FIG. 1. Simulation Chaotic Map

One important feature of this model must be noted: it is not strictly chaotic. Figure 1, is actually a plot of 10,000 drops. Unlike the physical chaotic attractor, or the attractor of some other maps such as the logistic map, we see that this attractor is actually one of very high periodicity. This can be deduced visually from the small number of visible points compared to the number plotted. It was determined in MATLAB that this model actually exhibits a 66-period motion at the $v_0 = 0.115$ value quoted by Kiyono and Fuchikami to be a region of chaos. However, one must also note that the model is extremely sensitive to slight changes to the flow rate in this region. A flow rate less than 0.1% different, $v_0 = 0.1149$, exhibits a 12-period, and $v_0 = 0.1151$ exhibits a 14-period. In any physical system, there will be slight variations in the flow rate which would cause the period of the motion to change faster than a full period could be completed. Thus, the system would likely appear to be truly chaotic, even if it were fully described by this, non-chaotic, mass-spring model.

Finally, a bifurcation diagram was produced from a $v_0 = 0.04$ to $v_0 = 0.11$. One can clearly see that the overall trend is for the period to decrease. This is punctuated by periods of bifurcations to “chaotic” regimes, which then transition into a periodic window of period-two, and then another series of bifurcations to chaos and a collapse to single-

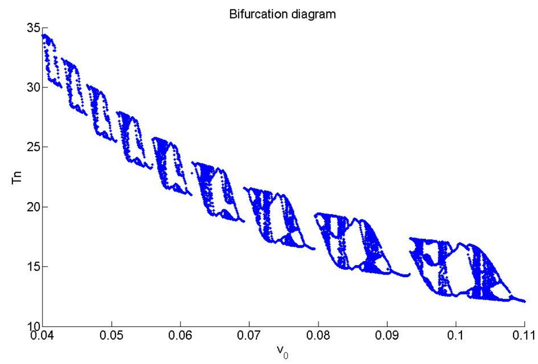

FIG. 2. Bifurcation Diagram

period. Each of these multi-period regimes is seen to be roughly symmetric. This behavior is attributed to the nature of our model's map. We can see in the return map for the chaotic case that the attractor intersects itself, which implies that the map is not unimodal. This is what produces the collapse of chaos to single-period, as discussed in Kiyono et. al.

IV. EXPERIMENTAL SETUP

The experimental setup went through a series of changes before an accurate, systematic method was converged upon. At first, the classical setup of a bucket with tubing through a hole in the bottom was used. However, since a stopcock was not readily available, very thin tubing was necessary to acheive the desired rate of droplet formation. The system was found to be highly sensitive to nozzle diameter, with smaller nozzle diameters causing lateral movement of the droplets due to the imperfections in the tubing. Additionally, it was very difficult to specify a desired flowrate.

Next, a syringe pump was used to mechanically control the flow rate. However, this was quickly abandoned due to drastic, mechanically-induced oscillations in flow rate. These oscillations visibilly affected the droplet formation, cycling back and forth between droplets and a steady stream. However, the ability to mechanically control the flow rate was considered a must have, so other mechanisms were sought after.

For the final experimental setup, a flow regulator and pump was used. Similar to the syringe pump, it allowed for specification of the flow rate but no oscillations were observed. The rest of the setup included a photodiode, laser, reservoir, and large diameter flexible tubing. The laser and photodiode were used to measure the period in between drops, while the reservoir collected the drops and served to supply the pump with fluid. The tubing was used to transport the fluid from the resevoir to the pump and to act as a nozzle for droplets to form on. The output of the photodiode was measured using an analog to digital converter, and fed into LabVIEW using a VI created by Nick Gravish.

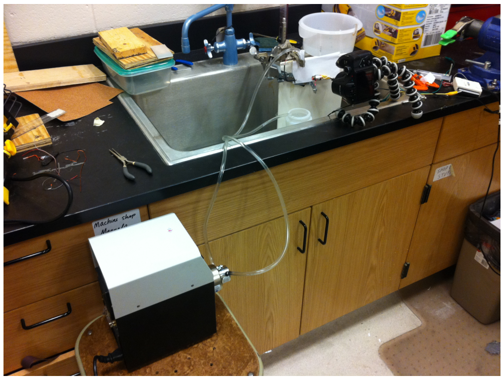

FIG. 3. Experimental Setup

FIG. 5. Closeup of the Flow Regulator Control

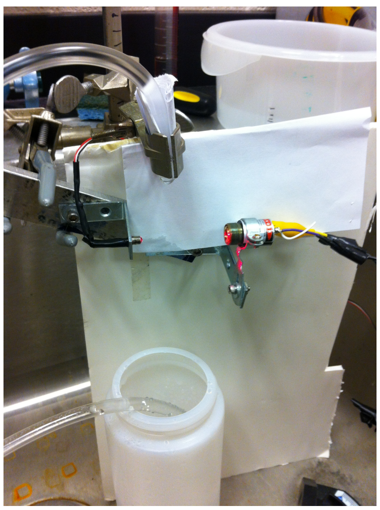

FIG. 4. Closeup of the Laser and Photodiode

V. DATA ANALYSIS

In order to obtain the period between drops from the raw data, some data analysis was performed. At first, the crossings between the raw data and a threshold value was used to find the downward spikes in the data, indicating a drop falling and blocking the light from the laser. However, there were two issues that were discovered and had to be accounted for. First, satellite drops were discovered to exist. These are drops that come off the main drop itself. This leads to incorrect period measurements as the satellite drops themselves are counted at main drops, which initially led to erroneous two, four and chaotic period data. This was corrected by lowering the laser crossing threshold. This removed the satellite drops which were inherently smaller than the main drops and blocked less light from the laser.

Second, when the drop fell through the laser, the top and bottom of the drop were double counted as crossings due to a phenomenon know as bouncing, often seen in mechanical contacts and switches. In order to remove this error, a refractory period was used to ignore threshold crossings immediately after an initial crossing. These changes were implemented during the data processing phase via MATLAB, and helped produce accurate secondary data that could be used to analyze the period doubling and chaotic regions.

Figure 6 shows the raw data obtained from a single drop. The double peak on the left indicates the bounced main drop while the smaller peak on the right indicates a satellite drop.

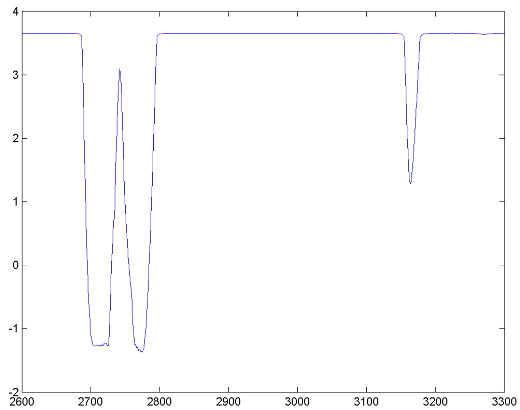

FIG. 6. Raw data for a Songle Drop

VI. RESULTS

An example of Period-1 behavior was found at a flow rate of 0.210 mL/s. Figure 7 is a Poincare map of next period (T_{n+1}) versus current period (T_n) with the line of symmetry drawn in green. The black dots, representing the data obtained, are all clustered in one location, about six seconds. Since there is only one cluster, and this cluster lies on the line of symmetry, a period of six seconds is a fixed point for the flow rate of 0.210 mL/s. This is shown in another way in Figure 8. Here period (T_n) is plotted versus drop count. The single horizontal line indicates that the period remains constant with each drop.

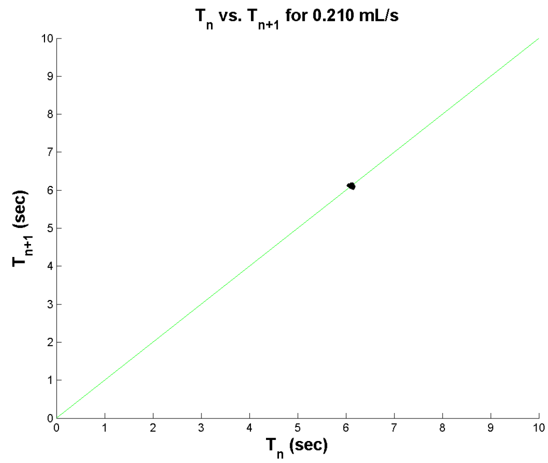

FIG. 7. Period-1 T_{n+1} vs T_n

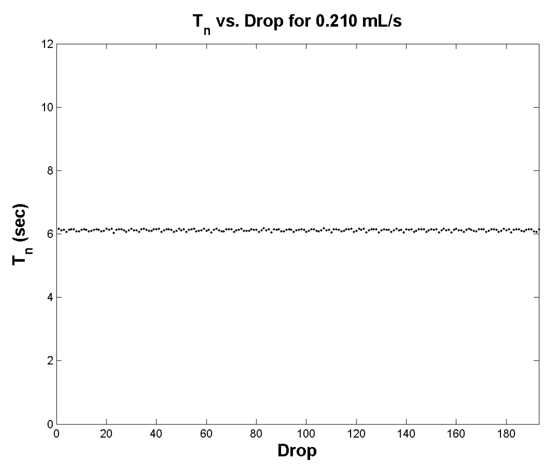

FIG. 8. Period-1 T_n vs n

Next, an example of Period-2 behavior was found at a flow rate of 0.319 mL/s. Once again there are two plots below showing the processed data: Figure 9, a Poincare map, and Figure 10, a time series. The Poincare map now exhibits two distinct clusters, one a little under four seconds and the other a little under four and a half seconds. This indicates that the period alternates between these two values. To make this clearer, consider the following line of events. A pair of drops falls with a period of four seconds, which if looking at the x-axis (T_n) places us in the upper left cluster of the map. Reading the y-axis (T_{n+1}) dictates that the next period will be four and a half seconds. Returning to the x-axis, we are now in the lower right cluster of the map. Finally, reading the y-axis dictates that the next period will be four seconds which returns us to our initial period. Figure 10 shows two horizontal lines representing the two alternating values of period.

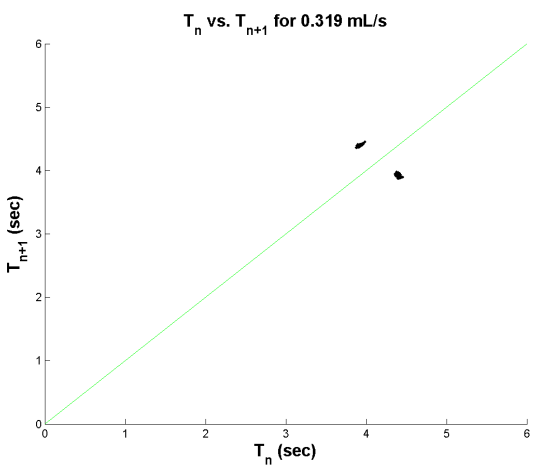

FIG. 9. Period-2 T_{n+1} vs T_n

Next, chaos was found in a range of flow rates. Figure 11 shows period versus drop count at a flow rate of 0.374 mL/s. Notice the seemingly random behavior. Although the range of periods appears to be bounded between about three and a half and four sec-

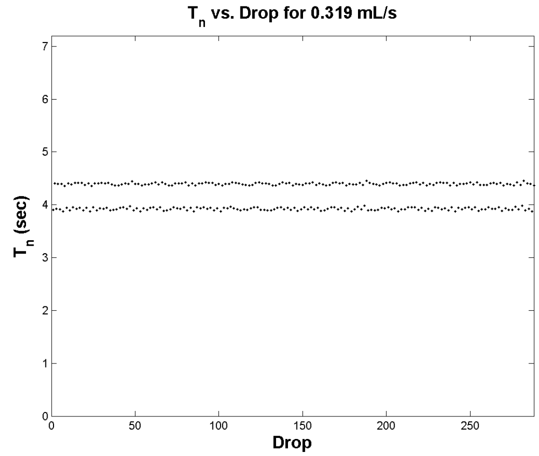

FIG. 10. Period-2 T_n vs n

onds, there does not seem to be any predictable pattern.

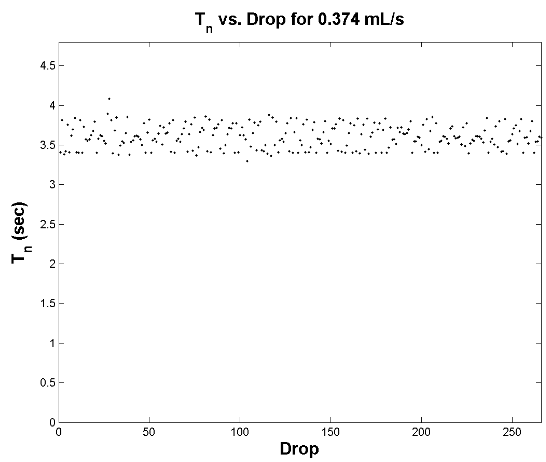

FIG. 11. Chaotic T_n vs n

However, in the Poincare map, beautiful attractors emerge. Figures 12 - 14 are three Poincare maps for three different flow rates. Even though the flow rates are different, notice the similarity in the attractors. They all seem to have an M shape. The plot on the far right was actually obtained from using a different nozzle diameter which would explain its slight difference in shape.

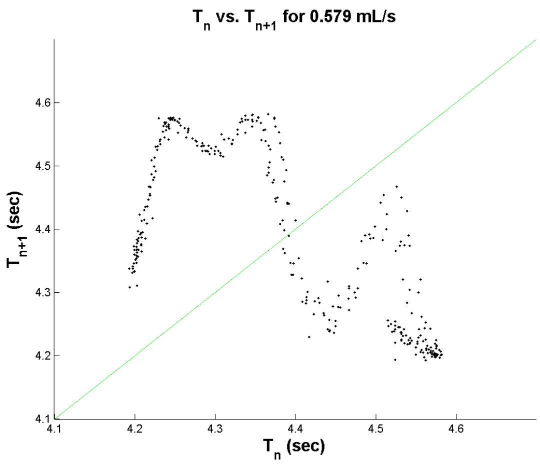

FIG. 12. Chaotic T_{n+1} vs T_n

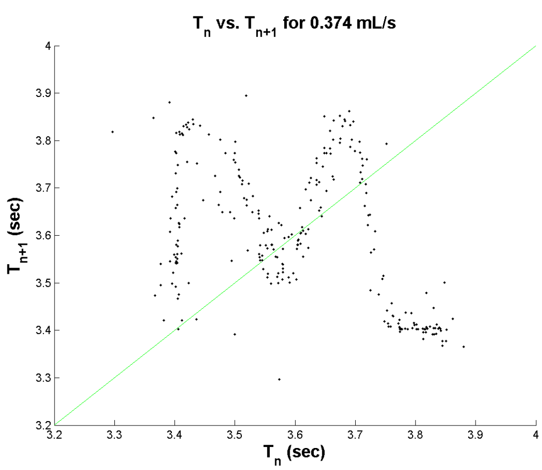

FIG. 13. Chaotic T_{n+1} vs T_n

Finally, an example of Period-3 behavior was found at a flow rate of 0.365 mL/s. Once again there are two plots below showing the processed data: Figure 15, a Poincare map, and Figure 16, a time series. The Poincare map now exhibits three clusters indicating that the period cycles between these three values. To make this clear, consider starting with a period of about 3.9 seconds, which

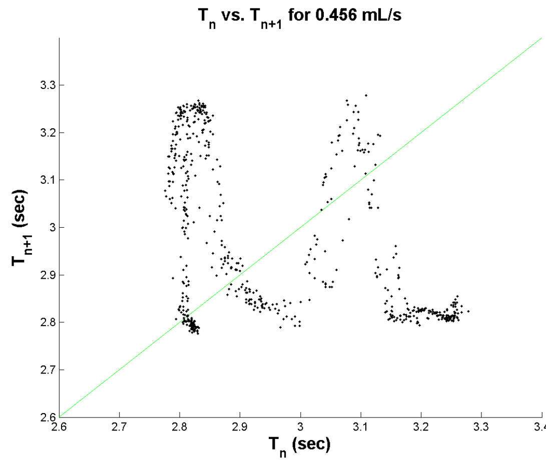

FIG. 14. Chaotic T_{n+1} vs T_n

places us in the bottom right cluster. The y-axis dictates the next period will be about 3.5 seconds. Returning to the x-axis, this places us in the far left cluster. The y-axis now dictates that the next period will be about 3.8 seconds. Once again, returning to the x-axis places us in the upper, middle cluster. Finally, the y-axis dictates a return to a period of 3.9 seconds.

However, compared to Period-1 and Period-2, there seems to be a fair amount of points lying outside of the three clusters. This can be explained by looking at Figure 16. There are regions where the three horizontal lines are distinct and then regions of apparent chaos. This is actually the system exhibiting transient chaos. Since Period-3 is a periodic window within the range of chaos, the system may sway both ways if it is on the border. This plot clearly indicates a swaying between Period-3 behavior and chaotic behavior.

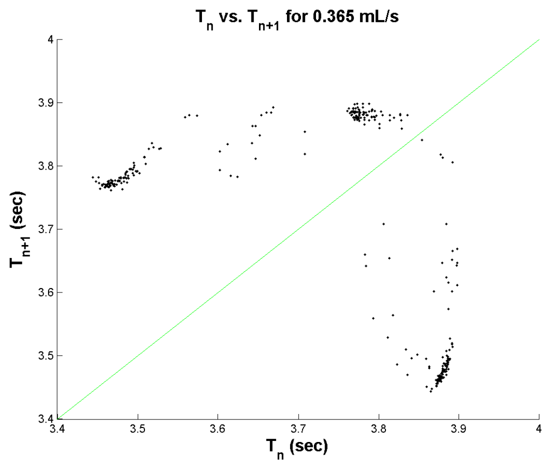

FIG. 15. Period-3 T_{n+1} vs T_n

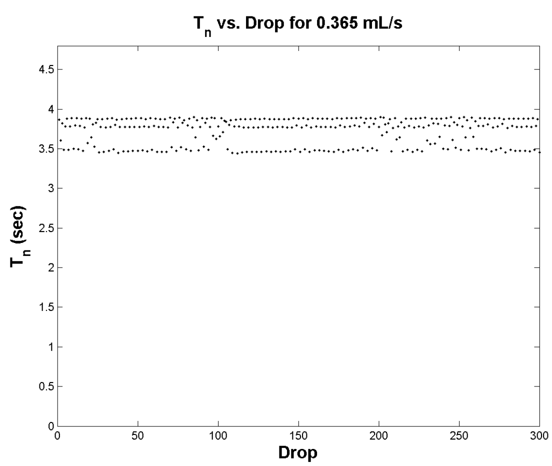

FIG. 16. Period-3 T_n vs n

VII. DISCUSSION

The type of period doubling and chaotic behavior shown in the experimental data qualitatively matches data in the literature[4], as well as the data produced by the model. We can certainly see period doubling, chaos, all matching between simulation and experimental data and reference data from Dreyer and Hickey. There is the region of period doubling, where a consistent period bifurcates into two consistent but different periods. This single and period-2 data has been shown in the experimental results, and confirmed by this qualitatively matches the reference data. We can also see period-3 data matching the reference data.

Due to the limited resolution of the pump with the current configuration, it was not possible to take data with fine enough flow regulation to reconstruct a bifurcation diagram experimentally. However, given the type of bifurcation data produced from the simulation, it is suspected that the map is in fact not unimodal, and does not follow the U-sequence exactly. A discussion of the U-sequence and unimodal maps is given in Strogatz. Though trends are known for this system, research remains to characterize the actual bifurcation parameters of the chaotic dripping faucet.

In addition to limited flow regulator resolution, it is suspected that pump vibrations and mode interactions contributed to the difficulty reproducing period-4 data. The mechanical oscillations of the syringe pump were visible to the naked eye, but not with the final setup of pump and flow regulator. Still, patterns on a very small scale formed before period doubling with the pump, indicating some external disturbance. It is suspected that this external disturbance is induced by oscillations in the flow, due to mechanics of the flow regulator. A sinusoidal disturbance such as this could be the reason that finer regions of the bifurcation diagram could not be investigated. That is, if there is a sinusoidal disturbance on the input variable, it could cause fluctuation over the period-2/period-4/chaotic region if these regions are sufficiently narrow.

VIII. CONCLUSIONS

The data are found to follow understood period-doubling and chaotic behavior. However, additional modeling and experimental work remains to fully understand the system. Recent research has proposed modified models of the system, which seem to predict data well. However, given the limited flow regulator resolution used in this experiment, it is not definitive that these models describe the dynamics of the system, as a unimodal map with a U-sequence of bifurcations could also have produced the data shown above. In short, the bifurcations of this system are still not completely understood, nor are the dynamical models completely resolved.

The lessons learned through this research should include fine attention to detail regarding the experimental setup. It is imperative to have accurate flow control and uniform flow rate. It is suggested that shunting the flow to allow finer resolution of the water passing through the nozzle, while still using the crude units of the pump, be performed in order to increase the flow rate resolution. This would allow finer investigation of the 70-90 flow-rate unit (FRU) region of the pump, where the period doubling and chaotic behavior seemed to be most apparent. Furthermore, large nozzle size is important to minimize imperfections and allow harmonic oscillator dynamics. For data collection, care must be taken to exclude satellite drops and debounce the laser-photodiode to remove the double crossings generated by falling drops.

What has been shown is that two routes to chaos appear to be confirmed, as predicted by Dreyer and Hickey. This is seen through the experimental and simulated data showing period doubling, as well as the periodic windows and transient chaos. This qualitative match between data suggest that further research be conducted to produce quantitative, and finer resolution comparisons. Overall, the model and secondary data match the literature and simulation qualitatively.

REFERENCES

[1]P. S. P. Martien, S.C. Pope and R. Shaw, "The chaotic behavior of the leaky faucet," Physics Letters A **110**, 399–404 (1985).

[2]K. Kiyono and N. Fuchikami, "Dripping faucet dynamics by an improved mass-spring model," Journal of the Physical Society of Japan **68**, 3259–3270 (1985).

[3]L. M. P. Coullet and C. Riera, "Hydrodynamical models for the chaotic dripping faucet," Journal of Fluid Mechanics **526**, 1–17 (2005).

[4]K. Dreyer and F. Hickey, "The route to chaos in a dripping water faucet," American Journal of Physics **59**, 619–627 (1991).