

An analysis of how coupling parameters influence nonlinear oscillator synchronization

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(Dated: 14 December 2012)

The coupled nonlinear oscillator has long been a useful paradigm for analyzing the dynamics of entrainment and synchronization in biological systems. However, there exists a gap in this field of study, wherein little has been done to address how oscillator synchronization is influenced by spatial parameters. In this study, we attempt to simulate the dynamics of firefly synchronization with a Kuramoto-derived model, while examining the influence of firefly coupling strength and sight distance on synchronization behavior. Our results demonstrate a quantifiable average relationship between these coupling parameters and the system's synchronization time, even in the absence of defining specific initial conditions. Future directions for investigation are discussed as well.

I. INTRODUCTION

To date, the concept of coupled nonlinear oscillators has been applied towards modeling and describing the behavior of many biological systems, such as the pacemaker cells of the heart, neurons, and crickets.¹²³ Particularly breathtaking to behold is the synchronization of large populations of fireflies, a natural phenomena that has been the focus of many studies over the years. Various models have been proposed on how the different species of fireflies communicate and synchronize, and these strategies include "phase advance synchrony", "phase delay

synchrony", and "perfect synchrony".⁴ Phase advance synchrony is a mechanism in which the fireflies only advance their phase upon seeing a flash, a similar behavior to the integrate and fire model proposed by Mirollo and Strogatz.⁵ The opposing mechanism is the phase delay behavior, where fireflies delay their phase advance upon seeing a flash, allowing them to achieve zero phase lag synchronization if the stimulus is close to their natural frequency. Perfect synchrony, on the other hand, allows fireflies to synchronize at zero phase lag even outside their natural frequency, implying that they are capable of actively modulating their internal frequencies.

While the investigation of firefly synchronization mechanisms has been widespread, little has been done to assess the implications of spatial structure on synchronization, even for the general case of nonlinear coupled oscillators.⁵ Therefore, it was our goal to implement a spatial representation for a fixed population of fireflies and analyze the implications it had on the synchronization behavior and time scale.

II. METHODS

A. The Model

In selecting a model that prescribed our individual fireflies' behaviors, we decided upon utilizing a modified version of the Kuramoto model. The Kuramoto model is a mathematical model that describes oscillator synchronization and is traditionally written as follows:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\phi_j - \phi_i)$$

To better examine the group dynamics of the oscillators, we can utilize the following "order" parameters:

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^N e^{i\phi_j}$$

where r is a measure of the collective behavior of the system and ψ is the average phase of all of the oscillators.

Now divide both sides by $e^{i\phi_i}$ to get:

$$r e^{i(\psi - \phi_i)} = \frac{1}{N} \sum_{j=1}^N e^{i(\phi_j - \phi_i)}$$

where the imaginary parts are:

$$r \sin(\psi - \phi_i) = \frac{1}{N} \sum_{j=1}^N \sin(\phi_j - \phi_i)$$

When substituted into the Kuramoto model, this yields:

$$\frac{d\theta_i}{dt} = \omega_i + K * r * \sin(\psi - \phi_i)$$

In this form, we can see that the individual oscillators are effectively updating their individual phases continuously based on the mean phase of the entire oscillator population, a factor that contributes to the model's high synchronization rate.

However, the reality is that fireflies can only update their phase when they see another firefly fire, requiring for the model to be modified to better represent the biological behavior. Reframing the Kuramoto model in these terms gives:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j \in \text{fired}} \sin(\phi_j - \phi_i)$$

The new summation term can be perceived as including an indicator function, where only the phases of "fired" are included in the phase update of the firefly.

Suppose that the firefly has an internal voltage-like state variable, x_i that controls when the firefly fires, and is subject to perturbation, ϵ , in response to other fireflies firing. Assume $x = f(\phi)$, where $f : \phi \mapsto x$ is smooth, monotonically increasing, and concave down, i.e., $f' > 0$ and $f'' < 0$, furthermore:

$$(i) \frac{d\phi}{dt} = \frac{1}{T}$$

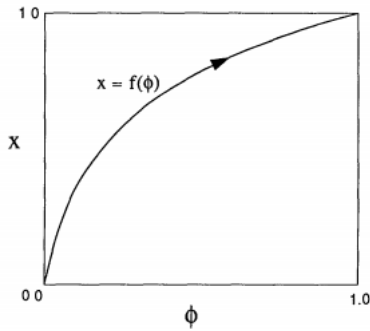
(ii) When the oscillator is at lowest state:

$$x = 0, \phi = 0$$

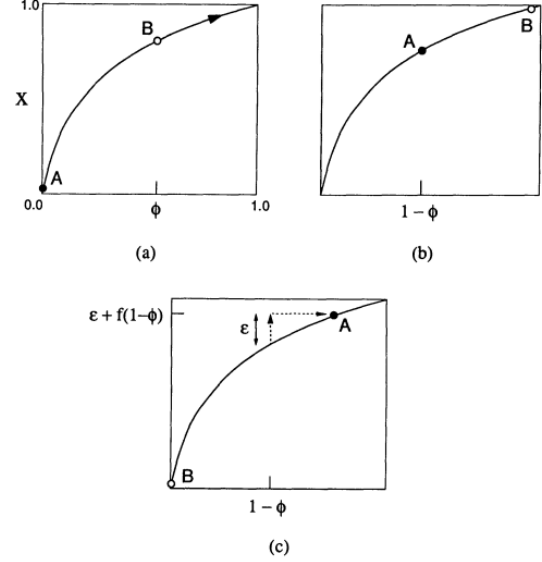
(iii) When the oscillator hits threshold:

$$x = 2\pi, \phi = 2\pi$$

Adhering to these definitions, we observe a diagram representation of a firefly's internal state as a function of its phase:



Taking a step further, we observe a two-oscillator system, in which firefly B fires in frame (b), causing firefly A to experience a state increase perturbation, which is the equivalent of driving its phase forward.



From this model design, we can make an interesting observation. Since a firefly's phase is only updated by the firing fireflies (which are at $\phi = 2\pi$) around it, the update function approximately becomes

$$\frac{d\theta_i}{dt} = \omega_t + \frac{K}{N} \sum_{j \in \text{fired}} -\sin(\phi_j)$$

In this form, we can see that when the internal phase of the firefly is more than π when it experiences a flash perturbation, it is phase-shifted forward. However, when the internal phase is less than π , the firefly is phase-shifted back, or experiences a "phase delay". In other words, the firefly is always choosing the strategy in which it will proceed in the direction of least phase shift from the firefly that fired. This indicates that our modified Kuramoto model exhibits the combined behavior of the two firefly synchrony mechanisms, "phase advance"

and "phase delay", and may be a closer analog to the "perfect" synchronization strategy observed in fireflies.

B. The Experimental Setup

For our experiment, we utilized the "Peggy" LED board, which utilized an Arduino microcontroller to communicate with the 625 LEDs that were available on the panel surface, of which we only used 48 in the form of a 6-by-8 LED array. Our model was prescribed in the form of a Matlab code, which consisted of the update function and the interface that allowed the computer to communicate with the Peggy board and dictate which LEDs to turn on or off. Additionally, a webcam was utilized to capture frames of the LED board in order to obtain through image processing in Matlab which "fireflies" had fired, or lit up. This information would then cycle through the Matlab update function, and according to which "internal states" had exceeded threshold, the signal would be sent to Peggy to light the next frame. Essentially, the system operated like an integrate and fire system, but with some code-based phase behavior that didn't cause the phase of the fireflies to scale exactly back to zero after firing.

The experiments were all conducted in complete darkness to avoid any effects exter-

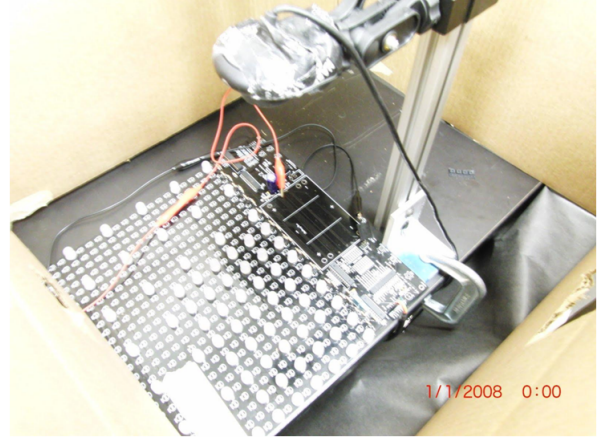


FIG. 1: Experimental setup

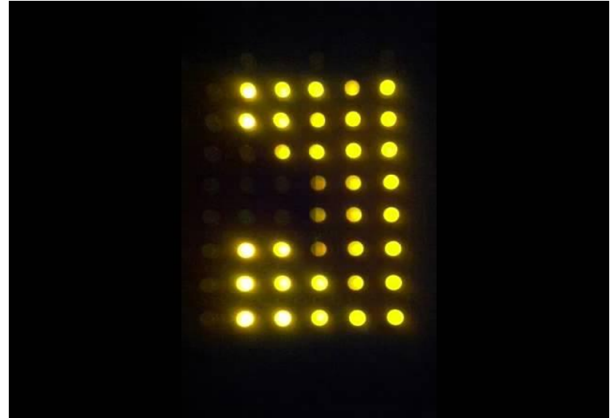


FIG. 2: Frame of an experimental trial

nal lighting may have had on the image processing calibration and each frame capture.

The synchronization parameters that were investigated were the global coupling strength and the number of connections per firefly. While the coupling strength was an easily modifiable multiplier within the Matlab code, the number of connections per firefly was more difficult parameter to set. The definition of a "connection" is such that if two fireflies are "connected", when one sees the

other flash, it will update its phase and vice-versa. If they are not connected, then one flashing won't yield any response from the other. Ultimately, the line-of-sight paradigm with repeating boundaries was selected due to its ability to maintain a homogeneous number of connections per firefly, offering a clear spatial representation without the uncertainty in the effects of varying connection densities. In fact, with repeating boundaries enforced on the 6-by-8 LED array, the connection scheme was comparable to having a lattice of connections on the surface of a torus, a potential 3-dimensional representation for firefly synchronization.

TABLE I: Synchronization Parameter Values

Coupling Strength(K)	Sight Radius/ Connections (C)
1	1/4
2	$\sqrt{2}/8$
5	2/12
7	$\sqrt{5}/20$
10	3/27
15	$\sqrt{13}/33$
20	4/40
25	$\sqrt{20}/46$
	5/47

To better illustrate how the sight radius (line-of-sight distance) translates to a certain

number of connections, Figure 3 and Figure 4 depict which connections are enclosed for what sight radii.

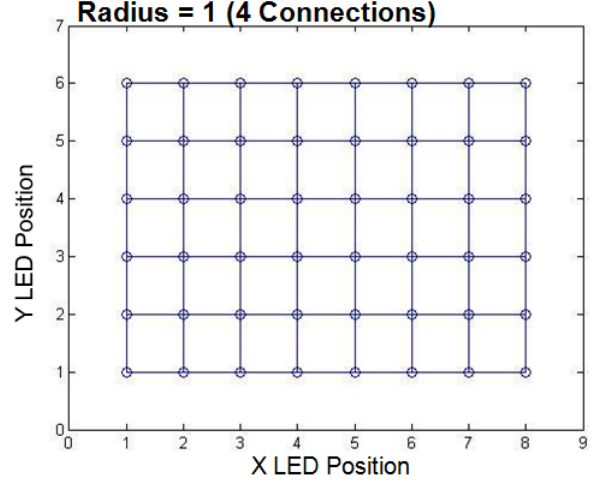


FIG. 3: Connections between LEDs

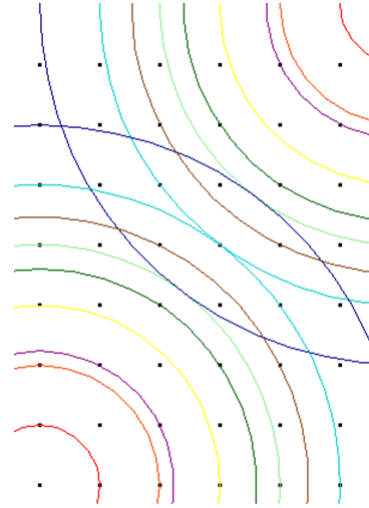


FIG. 4: Various sight radii

For each combinations of values in Table I, 15 trials were conducted, yielding 1080 trials total. At the beginning of each trial, the initial phases for each firefly were randomly generated according to a uniform distribution

between 0 and 2π . Additionally, the natural frequency of each firefly was held to be the same for all trials. During these trials, the number of "fireflies" that fired per loop iteration and the sum squared error of the internal phases per loop iteration were recorded and output as .mat files for analysis.

III. RESULTS

In Figure 5, the synchronization of fireflies given the same initial conditions but with different sight radii can be observed. The sight radius of $\sqrt{2}$ (8 connections/firefly) represented by the blue line appears to develop into a stable frequency as more and more fireflies are being entrained to fire simultaneously. Conversely, with the sight radius of 1 (4 connections/firefly), the red line does not appear to increase in number of fireflies fired simultaneously in a single frame/loop, and as a result, appears to have a less stable frequency. This outcome is logical, as one would expect as the number of connections/firefly increases, the synchronization time would decrease.

In Figure 6, we can again observe that the synchronization time for the system decreases as the sight radius decreases, where the synchronization time is defined as the loop iteration at which the sum squared error of the internal phases reaches 1, an error value

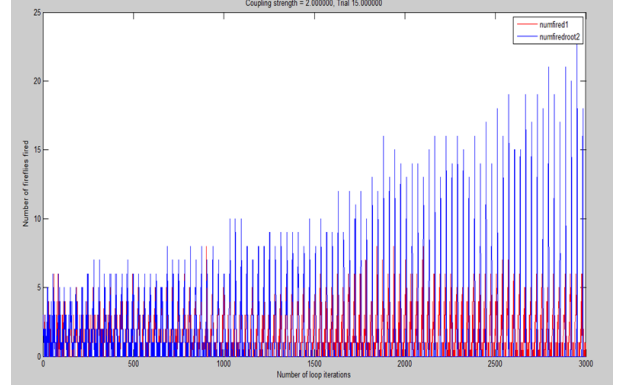


FIG. 5: Loop iteration vs. number fireflies fired

at which the lights consistently appear to be flashing in synchrony to the human eye. This plot is part of pilot data and is meant to illustrate the error metric used per trial to define system synchronization. Note also the inverse relationship between the sum squared error of the internal phases and the peak number of fireflies firing in the same frame (in synchrony).

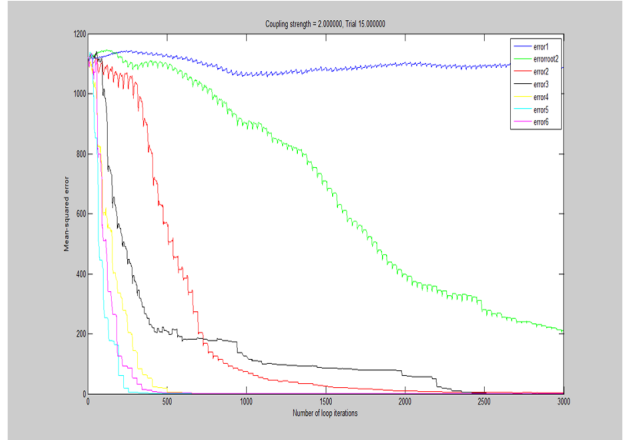


FIG. 6: Loop iteration vs. sum squared error of internal phases

In Figure 7, we can see that as the coupling strength of the system increases, the

convergence time for every case of connections per firefly decreases. Additionally, the convergence time for each repeated trial (marked by the same x-position x's) appears to be fairly clustered overall, justifying the use of a synchronization time average to estimate the system behavior given a coupling strength and a connection number. However, there appear to be clusters of trials for low coupling strength that fail to converge within our fixed trial time (4000 loop iterations), and will be removed from the data set during the regression in the discussion section.

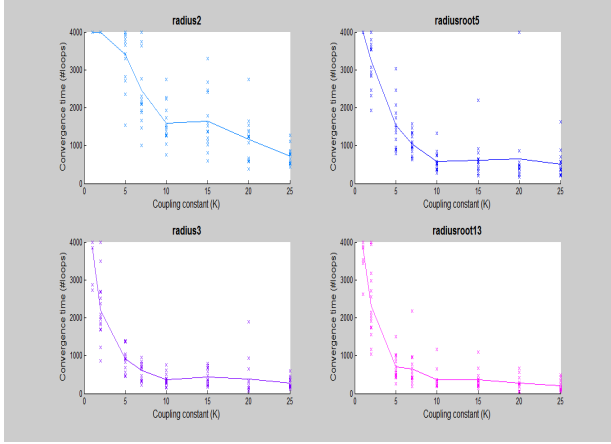


FIG. 7: Coupling strength vs. synchronization time by connection number

Superimposing each of these average synchronization times onto the same plot, Figure 8 demonstrates the consistent decrease in synchronization time for a given coupling strength as the connection number increases, which is a relationship defined more explicitly in Figure 9.

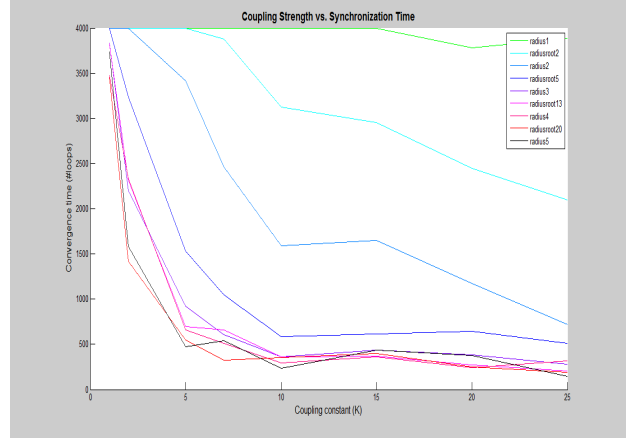


FIG. 8: Coupling strength vs. average synchronization time

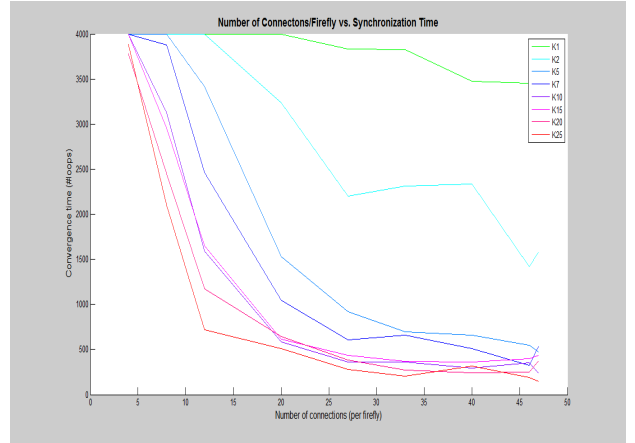


FIG. 9: Number of connections vs. average synchronization time

IV. DISCUSSION

In conducting this experiment, we have attempted to identify if there exists a potential relationship between the synchronization behavior of the system and the parameters of coupling strength and number of connections per firefly. The latter, being a parameter of spatial representation, is especially of inter-

est, and a regression of its relation to the average synchronization time is demonstrated in Figure 10. In said figure, the spread of data points per coupling strength appear to follow closely to a power law, except for the case of coupling strength $K = 1$. This inconsistency can potentially be ignored due the lack of data points at this coupling strength from removing non-convergent data.

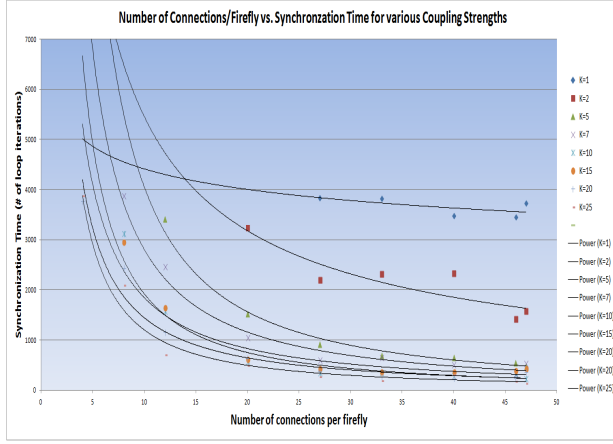


FIG. 10: Regression on C vs. average synchronization time

While the connection number versus average synchronization time relation appears to obey to a power law, the coupling strength versus average synchronization time relation adheres more closely to an exponential decay relation, as seen in Figure 11. It is from this fit that we will attempt to extract a preliminary equation that defines the synchronization time of our firefly system as a function of the coupling strength and the number of connections per firefly, or:

$$\tau_{synchronization} = f(\phi_0, K, C)$$

The ϕ_0 , or initial firefly phase distribution, is included in the equation due to its potentially substantial influence on the firefly synchronization time. This is apparent in the synchronization times of repeated trials (Figure 7) for different initial phases, but retaining the same coupling strength and connection number. While we approximated the range of initial phase influence by averaging across 15 different trials, there is certainly motivation to look into the consequences of different initial phase distributions. Furthermore, the properties of the initial phase distribution can be considered for both the distribution across phase space and physical space. Nonetheless, for the purposes of this study, the synchronization time is approximated to be a function of coupling strength and connection number only.

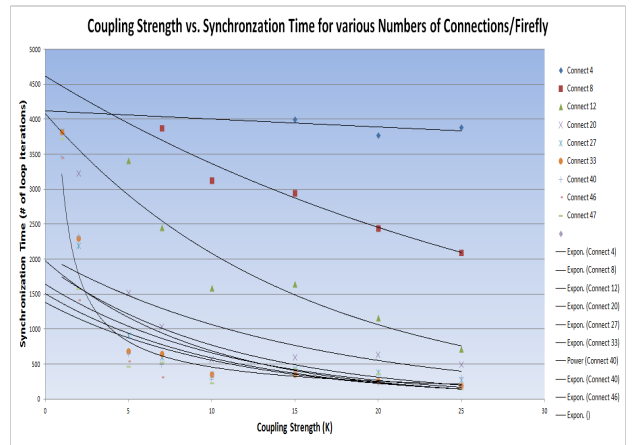


FIG. 11: Regression on K vs. average synchronization time

In fitting to the coupling strength versus synchronization time relation, we adopt the following regression equation form,

$$\tau_{synchronization} = ae^{bK}$$

where the coefficient "a" and "b" are defined as a function of the connection number, "C", by regression fitting. The defined relations can be seen in Figure 12 and Figure 13 below.

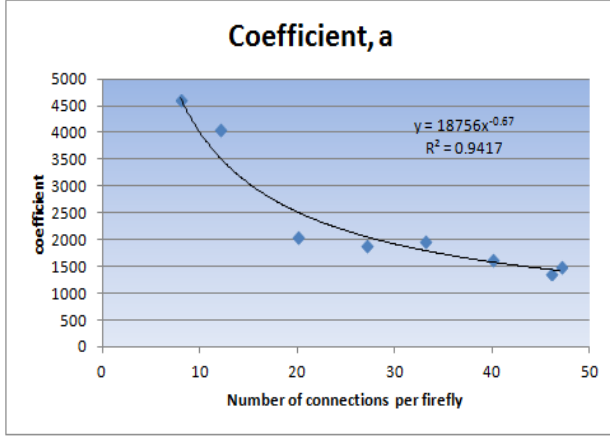


FIG. 12: Number of connections vs. coeff. a

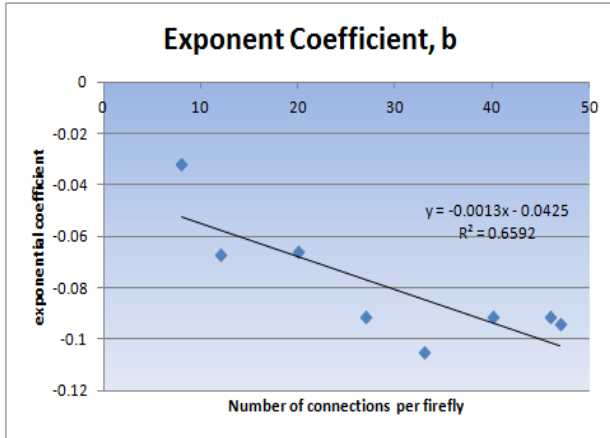


FIG. 13: Number of connections vs. coeff. b

By substituting in the coefficients in the exponential equation with the functions of the connection number found during regression, we obtain the following mathematical relation:

$$\tau_{synchronization} = \frac{18800}{C^{0.67}} e^{\frac{K}{1000}(1.3C+42.5)}$$

While the validity of this equation is likely far from accurate, this demonstrates the possibility of defining synchronization behavior, such as synchronization time, as a function of gain and spatial parameters, somewhat independently of the initial phase conditions. Of course, not enough initial phase cases have been investigated in this study to fully support this claim, which offers motivation for future studies in that direction. Ultimately, given the time to conduct a larger trial set and test the predictions of the synchronization time function against experimental results will help verify our current and future functions.

V. CONCLUSION

In this study, not many concrete conclusions can be drawn from the experimental results, as there are potentially too many synchronization behavior parameters that are being oversimplified by assumptions. However, the true gain from this brief research is in the methodology, in which two novel

approaches were developed. First was the modified Kuramoto model, which may be a new, untried method for describing synchronization in fireflies and other biological oscillators. Secondly, we addressed the effects of spatial parameters on synchronization behavior with the paradigm of line-of-sight, which allowed us to define a consistent metric for oscillator coupling connections from a spatial standpoint. By utilizing these two new approaches, it may be possible, even with the inclusion of physical barriers and noise, to more accurately describe biological synchronization behaviors in nature.

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