Magnetic plinko: exploring ski slope chaos

Andrew Masse,* Chris Cordell, and Andrew Hardin
School of Physics, Georgia Institute of Technology†
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I. INTRODUCTION

Inspired by the game Plinko featured on the popular American television gameshow The Price is Right™, we set out to model the chaotic behavior of a puck under the influence of gravity traveling through a lattice of plastic pegs.

FIG. 1. A contestant on The Price is Right™ tries her luck at Plinko. Little does she know, it is near impossible to control the outcome of the game.

While Plinko is essentially a pinball billiard game (partially elastic peg collisions and boundary interactions), it also demonstrates similar dynamics to classical billiard and pinball systems.

During our investigation of modeling the classic Plinko system, it quickly became apparent that solving the system computationally would prove very difficult.

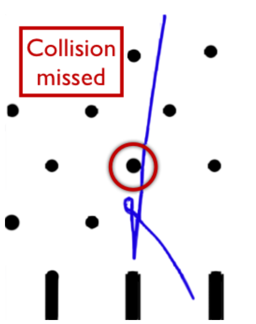

FIG. 2. An illustration of our numerical simulation missing collisions, a problem inherent in Matlab's solver.

With hopes of preserving the chaotic nature of the system, while providing a means for simulating the dynamics computationally, we created a new system involving multiple r^{-2} forces in a lattice in lieu of the plastic pegs of the Plinko board. Using simple and inexpensive materials, we were able to construct the aforementioned system using a lattice of rare Earth magnets interacting with a magnetic Plinko puck in an attracting configuration. This change necessitated obtaining theoretical predictions from a model which differed from classic billiard type systems.

II. THEORY

As our experiment shifted away from billiard mechanics, we entered another realm of chaotic systems. Rather than traveling through a lattice, deflected by discrete collisions with pegs, the strength of interactions in our new system is inversely proportional to the distance from the magnets, and continuous in time. Because the magnets are embedded in the Plinko board, the puck has an ability to pass over a magnet whereas the peg Plink system forbids the puck from passing through a peg. These differences change the dynamics of the system significantly, to the point where a billiard model is not the most effective way to analyze the system.

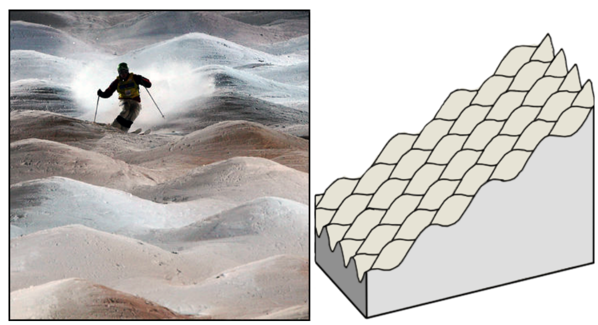

FIG. 3. (Left) A skier traverses a section of moguls, a real world Lorenz ski slope (Right) [1].

The new system we have devised resembles Lorenz's ski slope rather than a classic billiard or pinball system. In Lorenz's model [1], the ski slope is formed with regularly spaced humps as shown in FIG. 3. To simplify the problem, the system is reduced to a single particle sliding down the slope under the force of gravity with a constant coefficient of friction μ_k, and a variable normal force that keeps the particle from sinking into the slope or leaving the slope and becoming airborne.

Lorenz observed both a high sensitivity on initial conditions and the presence of an attractor. The attractor began to emerge when he examined the x position (on the horizontal axis in FIG. 4) and speed the speed of the particle for thousands of simulations. The result, shown in FIG. 5, reveals the formation of a pattern of "no recognizable form."[1]. In other words, the system is chaotic. Further analysis revealed the presence of a 3 dimensional attractor in the system.

The number of runs necessary to achieve similar structure to FIG. 5 in our magnetic Plinko experiment would be far too large to reproduce in a physical experiment, given

* amasse@gmail.com
† http://nldlab.gatech.edu/w/index.php?title=Group˙2

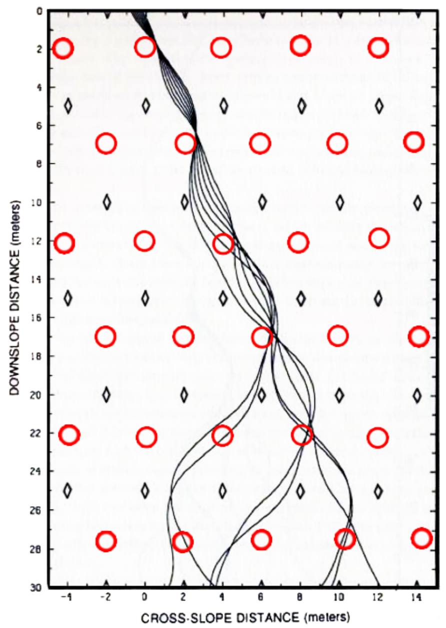

FIG. 4. Trajectories calculated by Lorenz for a particle traveling down a smoothly humped slope[1]. The black diamonds indicate the top of the slopes, while the red circles overlaid indicate the lowest point between humps.

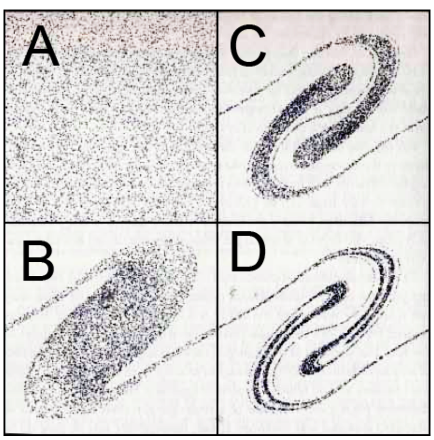

FIG. 5. (A) 5000 randomly chosen initial conditions for Lorenz's ski slope model. The horizontal represents the horizontal position on the slope, the vertical axis is the initial speed. Lorenz calculated these values after 5(B), 10(C), and 15(D) meters down the slope. [1]

the time constraints for data collection. We set out to collect the reasonable amount of experimental data necessary to both validate our simulation and determine reasonable parameters for the equations of motion to be used.

III. EXPERIMENT

Our experimental setup was optimized for rapid and efficient data collection. Despite knowing the number of experimental runs would be insufficient for examining the system at the level of detailed required, we wanted to obtain enough data to accurately tune our simulation using experimental parameters, and compare the trajectories observed to those of the Lorenz ski slope.

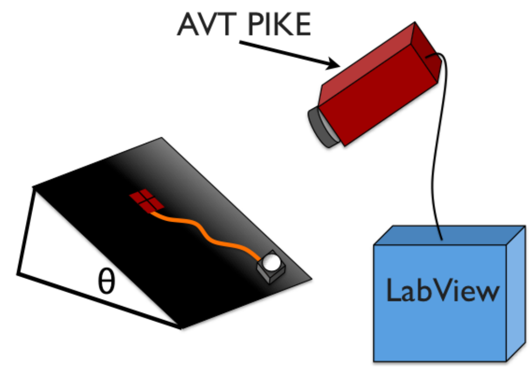

FIG. 6. A notional diagram of our experimental setup. θ, the slope angle used was 18°. See Table 1 for a complete list of experimental parameters.

Trajectories were captured using an Allied Vision Technologies ®PIKE camera. Each run was captured at 200 frames per second and transferred via IEEE1394b protocol at 800 Mbit/s. The fast data transfer rate allowed us to capture the trajectories in great detail. The puck, was fitted with a 15 mm diameter white sphere. The sphere was used to tracking in contrast with the black slope and backdrop.

Parameter	Value	Description
θ	18°	Angle of Plinko slope
$\mu_{kinetic}$	0.15	Coefficient of kinetic friction
N	28	Number of magnets

Table 1: Experimental Parameters

In order to speed up the data collection process we did point tracking directly in LabView. Not only did this allow us to track data real-time, but it helped avoid tracking raw videos of individual runs later on. In a relatively short amount of time, we were able to successfully track 400 trials, 100 from each corner of the red start box shown in FIG. III. We biased our starting locations to one side since the system should be symmetrical.

Qualitatively, our trajectories plotted in FIG. 7 look similar to the trajectories of the Lorenz ski slope, FIG. 4, when using the pit locations as attracting magnets. The systems are very different, yet the dynamics seem to exhibit similar characteristics.

IV. SIMULATION

After defining parameters experimentally, we began to numerically model our magnetic Plinko system. The first step was obtaining the equations of motion. We start by finding all of the different forces acting on the puck as it slides down the slope.

The force between two moving magnetic dipoles can be difficult and computationally expensive to model [2]. Instead we modeled the system by treating all magnets as

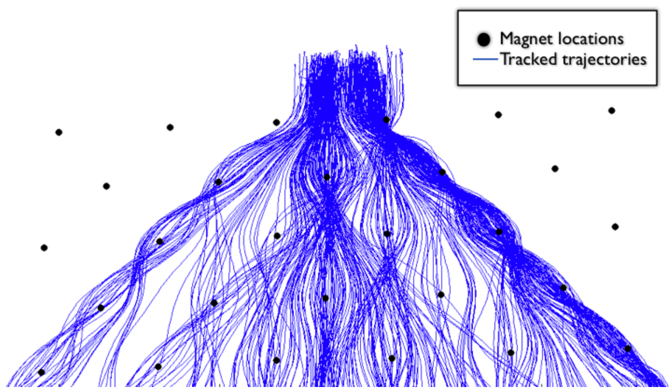

FIG. 7. Full set of trajectories as tracked in our experiment. Our initial x locations were biased slightly to the right.

magnetic monopoles of opposite polarity. By making such assumptions we arrive at the force model:

$$F = \frac{\mu_0 \cdot q_{m1} \cdot q_{m2}}{4\pi R^2} \quad (1)$$

Where μ_0 is the permeability of free space, and q_{m1} and q_{m2} are the magnitudes of the magnetic poles.

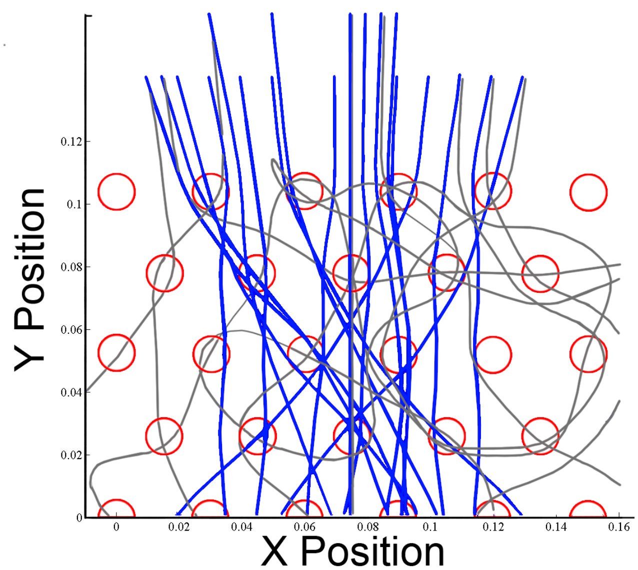

FIG. 8. Trajectories comparing r^{-2} (blue) with r^{-3} (gray). The r^{-3} does not correlate well with our physical experiment.

Since, in our system, the magnitude of the magnetic poles are the same for all magnets in the lattice, we simply use k in place of $(\mu q_{m1} q_{m2})/4\pi$. We determined k by running a full sweep of half of the Plinko board varying k and mapping out the final locations after a single interaction with the first row of magnets.

As FIG. 9 shows, above a threshold value, increasing k has little effect on the final position after passing through the first row of magnets. A k of $-.005$ generated trajectories very similar to our experiment and was used for the remainder of the simulations. Other k values were tested with less success. As a further test of our model, we ran

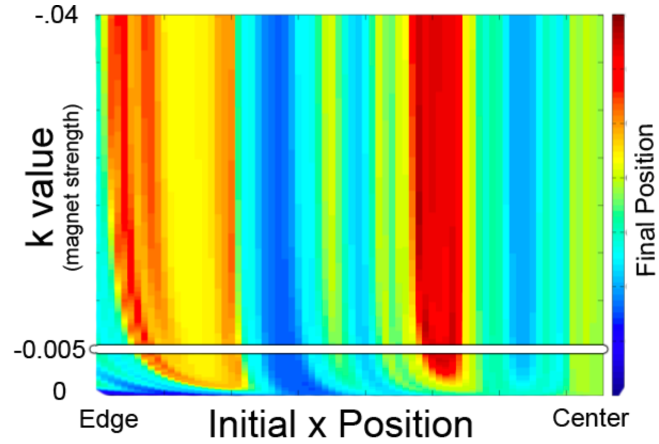

FIG. 9. Determining a reasonable k-value. The white line at -0.005 represents the value used for our simulations.

simulations using an r^{-3} force. The results shown in FIG. 8.

Using the experimental parameters listed in Table 1, we arrive at the following equations of motion to be used for our simulation.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ a_{g,x} + a_{f,x} + a_{m,x} \\ a_{g,y} + a_{f,y} + a_{m,y} \end{bmatrix} \quad (2)$$

Where $\vec{a_g}$ is given by

$$\vec{a}_{gravity} = [0, -g \cdot \sin\theta] \quad (3)$$

and θ is the tilt angle of the slope. The acceleration due to friction, $\vec{a_f}$, is given by

$$\vec{a}_{friction} = -\mu \cdot g \cdot \cos\theta \left[\frac{V_x}{V}, \frac{V_y}{V}\right] \quad (4)$$

The acceleration of the puck due to interactions with magnets in the lattice is modeled as

$$\vec{a}_{magnetic} = \sum_{i=1}^{N_{magnets}} \left(\frac{k}{{R_i}^2}\right) \cdot \hat{\boldsymbol{\imath}}_i \quad (5)$$

where

$$\vec{R} = [X - X_{magnet}, Y - Y_{magnet}] \quad (6)$$

and $\hat{\boldsymbol{\imath}}_i \equiv \frac{\vec{R}_i}{R}$.

With the equations of motions defined, we then used ODE45 solver in Matlab to generate trajectories sweeping initial x position and initial y velocity. A complete mapping of all initial and final locations (FIG. 10.

As we increase the initial drop height, the puck passes through the lattice at a higher speeds, and the trajectories are less perturbed by the magnetic forces. Clear basins of attraction seem to exist in regions where the puck passes close to a magnet. This set of conditions produces a similar final state, in agreement with our experimental observations, where the puck often followed a diagonal path as seen in FIG. 7.

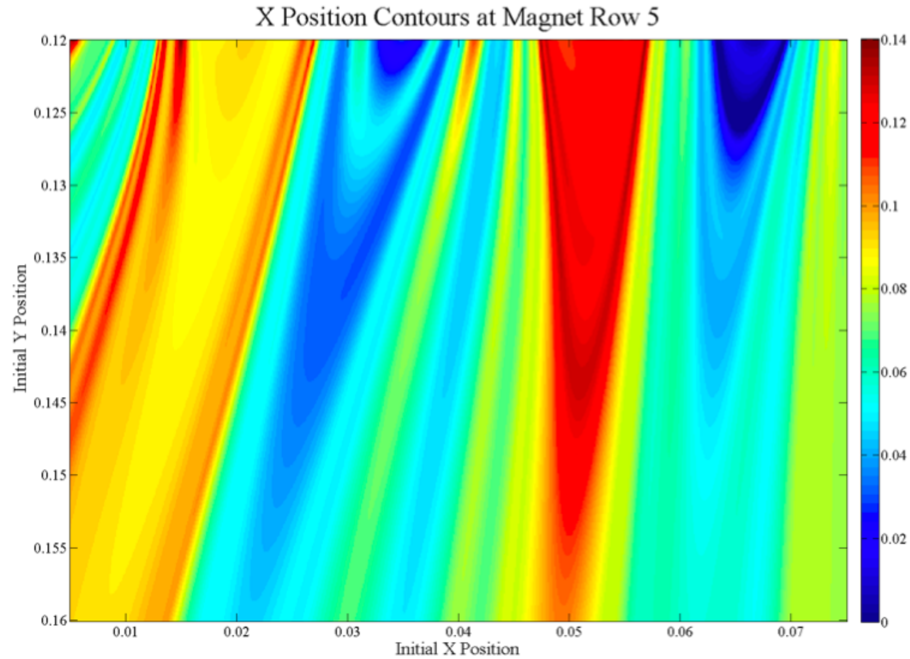

FIG. 10. A mapping of final cross-slope position for a range of initial conditions. Here we show one half of the Plinko board with initial x positions ranging from 0.5 cm to 7.5 cm (the center of the board), and initial y positions varied from 12 to 16 cm. Note: the mapping is symmetric about the center.

V. RESULTS AND DISCUSSION

A. Simulation results

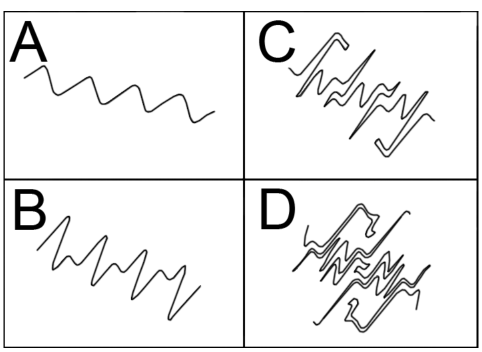

FIG. 11. In simulation we map the cross slope speed of the puck for a set of initial conditions. The horizontal axis represents the horizontal position on the slope, the vertical axis is the horizontal cross slope velocity after the puck passes rows 1(A), 2 (B), 3 (C), and 4(D).

Our numerical simulation allowed us to model the system in much greater detail than our physical experiment. Because the dynamics of our system is qualitatively similar to the Lorenz ski slope, we will attempt to reconstruct an attractor in a similar fashion. The tens of thousands of extra trials allow us to delve deep into the system in search of the chaotic dynamics illustrated in Lorenz's ski slope model [1]. In order to direct the system, we reduced the phase space in two ways to reconstruct an attractor.

In the first method, a set of initial x conditions was selected that spanned a portion of the cross-slope. We calculated the scalar puck speed as it passed successive rows of magnets. In FIG. 12 we plot the puck speed for varying cross slope x positions as the puck passed through the magnet lattice using a three dimensional technique used in *The Essence of Chaos*[1].

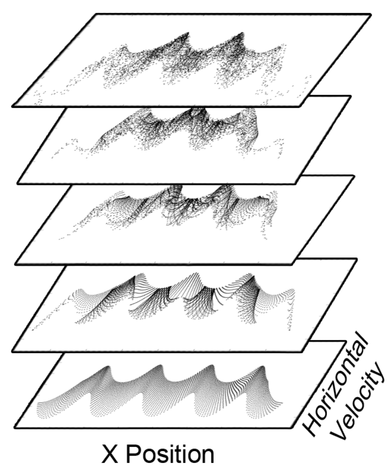

FIG. 12. A three dimensional view of the Plinko attractor. The top layer is after the fifth row of magnets.

The second method we used was to select only points with the same initial y values and examine the cross-slope velocity, $\dot{x}$. This result, shown in FIG. 11, is analogous to the ski slope attractor examined by Lorenz [1] (FIG. **??**) in that it shows clear folding and contraction of the phase space. There also appears to be a time evolution sensitive to starting position. It can be seen that as the system evolves, in FIG. 11 (C) for example, breaks in the initial symmetry occur, where very small deviations in initial conditions can cause the puck to reverse its horizontal velocity.

B. Experimental Results

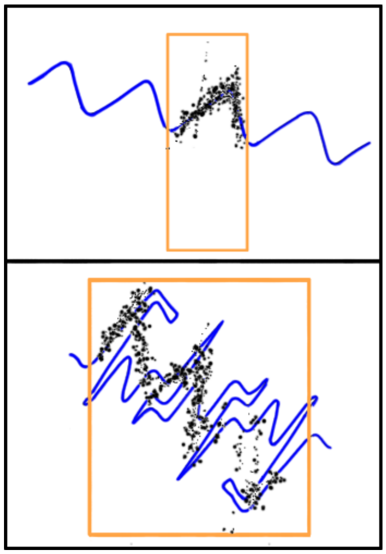

FIG. 13. Cross slope velocity, $\dot{x}$, of the puck plotted against cross slope position, x. The blue line is the simulation results, as shown in the previous section. Experimental results are overlaid in black. The orange rectangle indicates roughly the region of the simulation that matched the initial conditions tested experimentally.

As further verification of the accuracy, or at least plausibility, of our numerical simulation, we attempt to reconstruct an attractor from our experimental data. When we compare our experimental data directly to the simulation, similarities emerge. one can observe similarities in the shape of the cross slope velocity $\dot{x}$ and the cross slope position x. As mentioned in the experiment section, the angle of the camera used to record the data was not perfectly aligned. After proper scaling, examining $\dot{x}$ against x shows some agreement with the simulation as seen in FIG. 13. Here we have examined the cross slope velocity after the puck passed row one, and again after the puck passed row four. Nevertheless, agreement between simulation and experiment not only demonstrate our simulation is valid for some real parameters, but also the chaotic folding seen in great detail in the simulation, should also exist in the real experiment provided a denser set of data was acquired.

VI. DISCUSSION

During the timeframe allotted for this experiment, we were able to successfully develop a numerical simulation that models a real system and reproduces the rich chaotic dynamics inherent therein. Starting from Lorenz's ski slope model, we developed a magnetic Plinko system with sensitive dependence on initial conditions. Attracting magnets acted as the pits of the slope, deflecting a puck sliding down the slope with a low coefficient of friction. These deflections are analogous to a the deflection of a sled or tube as it passes through a section of regularly spaced moguls on a ski slope. Over the course of tens of thousands of simulations, we discovered an attractor in the system and mapped the folding of this attractor as the puck progressed down the slope. With the amount of experimental data collected, we were able to see traces of this attractor when examining the puck velocities against horizontal position after passing through each row of magnets in the slope.

The next step in an experiment of this nature would be to collect thousands upon thousands of experimental trials, sweeping a larger range of initial conditions. This would allow for the clear emergence of an attractor from experimental data, rather than relying on the accuracy of a simulation to produce results.

VII. ACKNOWLEDGEMENTS

We would like to thank Dr. Daniel Goldman for teaching the course (PHYS 4267) and providing helpful discussion regarding both our experiment and simulation. We would also like to thank the teaching assistant, Nick Gravish for his assistance in the experimental setup, and suggestions for data analysis. Proper guidance was helpful in obtaining interesting results. Finally we would like to thank the Complex Rheology and Biomechanics lab at Georgia Tech for allowing use of their laboratory space and supplies.

[1] E.Lorenz. *The essence of chaos*. Jessie and John Danz lectures. UCL Press, 1995.
[2] Griffiths, David J. *Introduction to Electrodynamics (3rd ed.)*. Prentice Hall, 1998.