Chaotic dynamics of a ski slope with a magnetic mogul field

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I. INTRODUCTION

The game of Plinko, as seen on the CBS television show The Price Is Right, demonstrates rich chaotic dynamics as evidenced by a strong sensitivity to initial conditions. This behavior can be readily observed by watching repeated runs of the Plinko puck and noting the final location of the puck once it has passed through the peg lattice: despite the fact that many runs often start nearly co-located, the runs will often take very different paths and the puck will find its way to a seemingly random final position. This observation leads us to believe that the system is indeed chaotic, exhibiting behavior similar to that of a pinball machine with dissipation or a vertical billiards table.

A review of the literature on these topics quickly reveals that the pinball machine offers rich chaotic dynamics when combined with a collision rule that adds energy into the system[1] (despite conventional notions that only dissipative systems show the presence of attracting behavior and phase space contraction[2]) and that pinball billiards shows equally ordered chaotic behavior when paired with a dissipative (non-elastic) collision rule and modified boundary properties[3]. These reviews led to the assumption that the chaotic behavior of the analogous systems would generalize well to the Plinko problem and that this would enable the discovery and mapping of basins of attraction and strange attractors within the system under study.

This paper discusses the issues with modeling Plinko and introduces an analogous model that was created from attempting to solve the problems related to Plinko's dynamics. This analog model was studied in place of the real Plinko system, and it's behavior compared to that of a closely related pedagogical example: ski slope chaos.

II. SYSTEM MODELING

Plinko Piecewise Model

In order to study this system, both a simulation and a physical experiment were built. The simulation was attempted in two ways: using a Runge-Kutta integration method to solve the puck's motion due to gravity with an event detection loop to detect collisions between the puck and the pegs; and the calculation of parabolic trajectories between pegs, root solving the equation of motion for the collisions. For both methods, upon the detection of a collision, the model calculated the reflection angle of the puck about the peg normal at the collision point with a user-defined coefficient of restitution. From this angle, a new initial velocity was determined and the process was iterated from that point. This procedure continued until the puck crossed the lowest row of pegs on the Plinko board. The final position bin was calculated and returned as the simulation result.

Both methods had significant problems, however, appropriately detecting collisions. The Runge-Kutta integrator often set the time step so high as to miss collisions (due to the simple nature of the equations of motion). The parabolic simulation was mostly immune to overstepping collisions at high (> 0.8) coefficients of restitution, but for values of the coefficient low enough to provide the dissipation necessary for the onset of chaos, even the parabolic simulation had singularity issues that led to trajectories clipping the pegs.

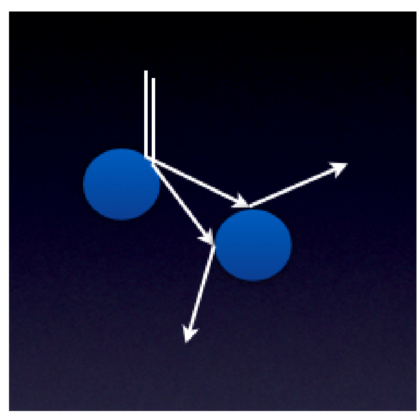

FIG. 1. In Plinko, two trajectories that start arbitrarily close diverge exponentially after finite time with no tendency to attract.

In addition to simulation issues, a map of the initial position and velocity against the final cross-slope position revealed that there was no structure in the input space and that there were no basins of attraction apparent in the system. Furthermore, no analysis of the data was able to reveal an attracting structure. It was determined that this system did not produce chaotic behavior simi-

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lar to the pinball and billiard systems previously studied. The results were, instead, distinctly lacking in structure, and small perturbations yielded extremely divergent behavior as opposed to even weakly chaotic ergodic motion. This is due to the exponential divergence of two similar trajectories as they evolve in the system (FIG. 1), and because there is no attractor limiting (or dissipating) that divergent behavior, even quasiperiodic orbits (Hamiltonian chaos) can be ruled out.

This disappointing result lead to a modification in the system under test and the underlying models in order to demonstrate the type of chaotic behavior desired in the experiment.

Plinko Force Field Model and Ski Slope Chaos

In order to solve the problem of modeling the system's dynamics, the rigid pegs (discontinuities) were replaced with a "force field" meant to allow for a continuous, integrable system of equations suitable for non-stiff Runge-Kutta integration. The most obvious experimental analog for a force field is a magnet, but this meant that the "pegs" would not longer produce collisions; they would simply influence the trajectory of the puck, a fundamental shift from Plinko.

Dissipation was also necessary for the system to have attracting rather than ergodic behavior; in order to incorporation dissipation, the Plinko board was rotated off-vertical. The resulting slope provided a normal force suitable for incorporating Coulomb friction into the model. This was a non-trivial modification to the system under test which causes this model to be a poor analog for the real Plinko system due to the inclusion of a non-vertical slope and sliding friction. Instead, the model now more closely resembled that of a ski slope[4].

This realization was transformational for the investigation, and it was re-tooled to investigate the dynamics of a ski slope similar to that of Lorenz with a square lattice of magnets as opposed to a mogul-like surface. This modification also allowed for simple physical experimentation whereas the Lorenz system is prohibitively difficult to build and operate with the parameters that he uses. The investigation thus became an analysis of the chaotic behavior (attractors in the phase space, basins of attraction, long-term evolution) of a magnetic puck in a magnetic field, and a comparison between those results and the simplified model studied by Lorenz.

Magnetic Ski Slope Model

The resulting system of equations was thus based on three categories of forces: gravity (1), friction (2) and magnetic force (3), where θ is the angle of the slope above a horizontal plane, g is the gravitational force on Earth's surface, μ is the coefficient of kinetic friction, and k is a "magnetic force coefficient" (defined in Section III). The system of equations are derived for a Cartesian coordinate system:

$$\vec{a}_{gravity} = [0, -g \cdot \sin\theta] \tag{1}$$

$$\vec{a}_{friction} = -\mu \cdot g \cdot \cos\theta \left[\frac{V_x}{|V|}, \frac{V_y}{|V|} \right] \tag{2}$$

$$\begin{gathered} \vec{R} = [X - X_{magnet}, Y - Y_{magnet}] \\ \vec{a}_{magnetic} = \sum_{i=1}^{N_{magnets}} \left(\frac{k}{R^2} \right) \frac{\vec{R}}{|R|} \end{gathered} \tag{3}$$

Based on these force definitions, a system of equations (4) for the motion of the puck can be calculated for numerical integration using MATLAB's ODE45 function.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ a_{g,x} + a_{f,x} + a_{m,x} \\ a_{g,y} + a_{f,y} + a_{m,y} \end{bmatrix} \tag{4}$$

The simulation was created using this system of equations in MATLAB. The constants (k, μ and θ) were calibrated using experimental data to match the experimental setup (discussed further in Section III. The experimental results were then used to validate the simulation by running a large number of cases in the simulation similar to those performed experimentally and comparing the results to those found in the physical experiment. Validation of the simulation then allowed us to explore the phase space and long term dynamics of the system in the detailed and controlled environment of the simulation in order to produce many orders of magnitude more results (order 10^4-10^5 cases with total run times from 0.5-20 hours). The results of the comparison between the simulation and experimental data, as well as the comparison with Lorenz's results, is detailed in Section IV.

III. EXPERIMENTAL SETUP AND DATA

The overall design of the experiment (FIG. 2) called for the creation of a sloped surface with field of small rare Earth magnets arranged in a square lattice and aligned such that each magnet had the same pole normal to the slope surface (FIG. 3). The puck was to be a small object containing a similar magnet such that the magnetic poles could be easily reversed in order to switch between attraction and repulsion with the magnetic lattice.

The puck and surface of the slope were both to be black, and the puck was to have a small white dot suitable for tracking using a high speed camera and LabView centroid tracking software. The camera was aligned so as to look down on the board from above. The angular bias of the camera pitch and run slope rotation angles were

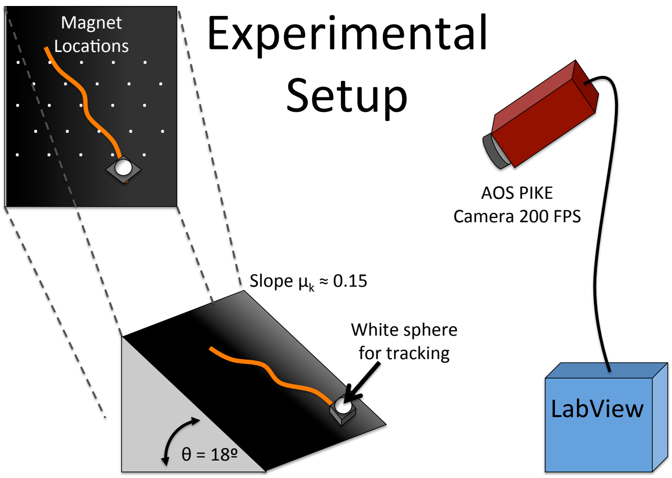

FIG. 2. This schematic shows the basic experimental setup and data collection system for the ski slop experiment.

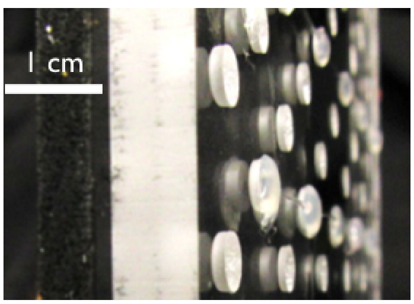

FIG. 3. A view of the plexiglass board into which the magnets were embedded which lies underneath the Teflon slope surface at a distance of 0.5 cm.

minimized prior to performing runs and were not changed during the course of the experiment though these values were non-zero and contribute to experimental data error.

Experiments were conducted by starting the puck in one of four regions near the cross-slope center line. Two regions were on the center line at slightly varying heights. The other two regions were slightly offset from the center line (to the right side) and at equal heights to the previous two regions. The puck was placed into each of the four regions and released without imparting initial velocity or spin.

Preliminary testing in simulation had shown some interesting dynamics in the repulsive case, however, the initial trials of the physical experiment showed that the puck was not stable (prone to flipping) in the repulsive configuration. The experiment and simulation were reconfigured to test the attracting condition which proved to be more stable. Additionally, in order to aid in tracking, a small plastic sphere was attached to the top of the puck (FIG. 4).

The experiment was executed 100 times per starting region using LabView to track the motion of the puck

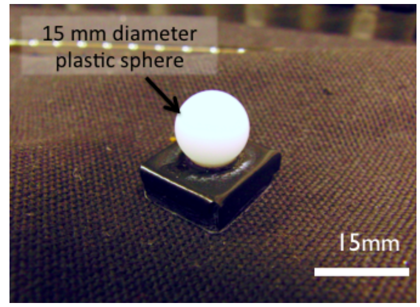

FIG. 4. The magnetic puck is made from a small Teflon block with a rare Earth dipole embedded into the bottom. A small white sphere is attached for tracking.

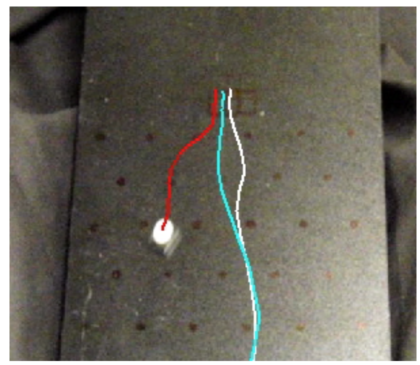

FIG. 5. Each trajectory was captured using the high speed video camera and LabView to track the puck's centroid. Several sample trajectories are shown here.

as it progressed. The resulting trajectories were saved (FIG. 6) for analysis and comparison to the expected results generated by the simulation.

In addition to collecting trajectory data, calibration data were collected for tuning simulation parameters. Among those: the coefficient of kinetic friction (μ) was measured to be 0.15; the slope of the board (θ) was set to 18° above horizontal, large enough to ensure that the puck did not stop but small enough to allow for the onset of chaos[4]; and the relative position of the magnets within the field and their spacing, 3 cm.

The magnetic force was calibrated using a qualitative analysis of the magnetic interactions in the experimental data and a simulation analysis of the effect of the magnetic force coefficient k on the resulting trajectories. Because the simulation was not designed to handle magnetic force by directly dealing with the magnetic fields of the dipole interactions (for speed and code simplicity), the Gilbert model was used.

This necessitated an evaluation of the end force instead of the magnetic fields of the magnets. In order to do this, the simulation was run for values of k ranging in magnitude from 0 to 0.04 (the sign of k is negative) from a single starting height and a range of cross-slope x positions. The results show (FIG. 7) that the puck drops straight to the bottom without deviation for sufficiently

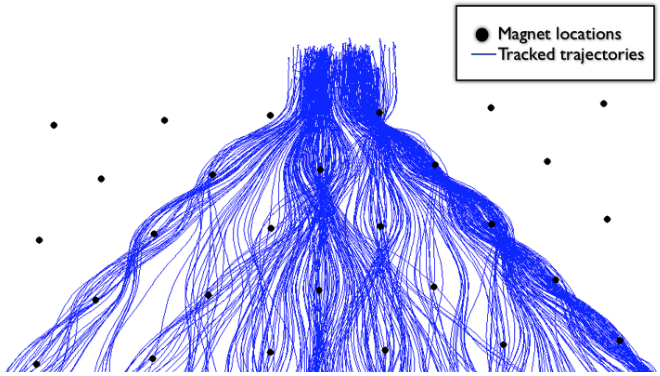

FIG. 6. All trajectories captured during the experiment, showing the path of the puck and the magnet locations.

small k values, a result which is not interesting from the standpoint of a chaotic analysis. Additionally, above a critical value k_{cr} the dynamics reach a "steady state" where they are no longer sensitive to changes in k. This critical value appears to be approximately $k = -0.005$, and this value was used thus selected for use in the simulation.

The trajectories that are generated in the simulation using this magnetic force coefficient (FIG. 8) are qualitatively similar to those found in the experimental data (FIG. 6), and are thus reassuring that the selected value of the coefficient and the magnetic force model are reasonable.

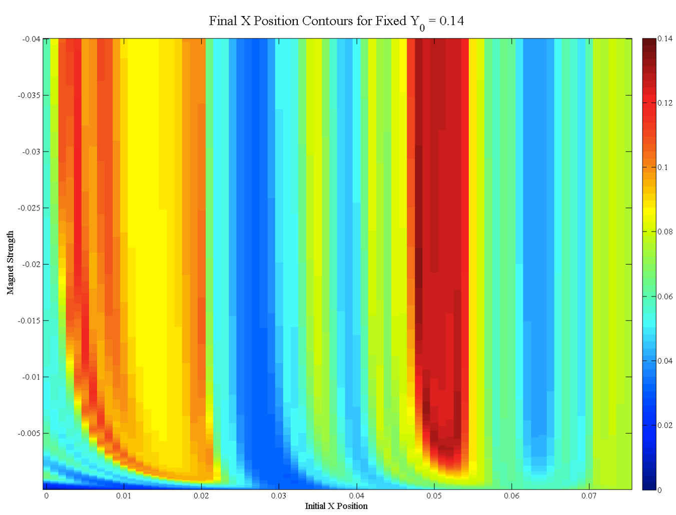

FIG. 7. A graph of magnetic force coefficient k vs. initial x_0 for a fixed $y_0 = 0.14$. This plot shows that around $k = -0.005$ the system is no longer sensitive to increased magnetic field strength.

It should be further noted that the true force between two dipoles has $\frac{1}{r^5}$ and $\frac{1}{r^3}$ interactions. However, the Gilbert model predicts the force interaction between

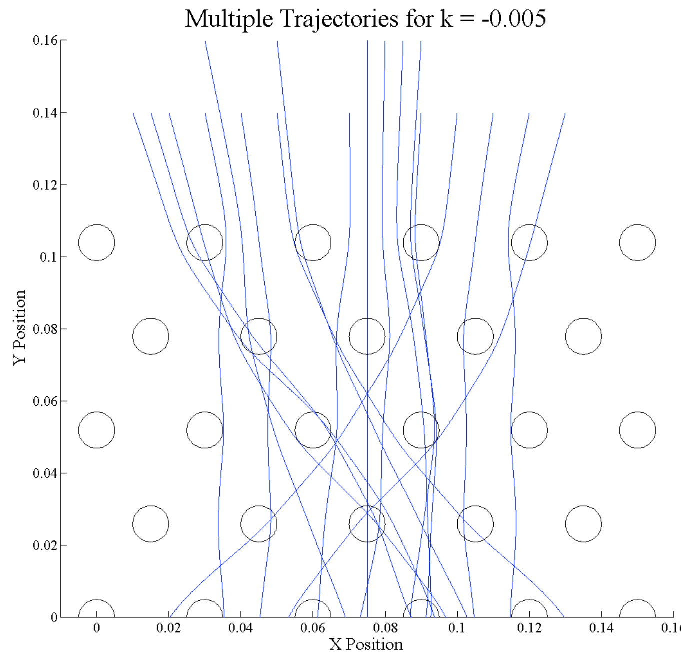

FIG. 8. Several sample trajectories run at $k = -0.005$ to show the behavior of the system relative to the expectations. These trajectories indicate the the choice of magnetic force coefficient produces physically reasonable results.

dipoles accurately for sufficiently small dipole magnets which seem to match the parameters of this experiment.

IV. RESULTS, ANALYSIS, AND DISCUSSION

Reconstructed Attractors

The phase space of the system was reduced in two ways in order to reconstruct the attractor. In the first method, a set of initial x conditions was selected that spanned a portion of the cross-slope with various initial x positions included in the set. The speed magnitude of the puck was then observed as it passed each row of magnets and the initial cross-slope x position was plotted against the speed magnitude at each row of magnets (FIG. 9 is an example taken at Row 2).

This shows an interesting albeit somewhat expected result: the experiment and the simulation show similar qualitative behavior in the region highlighted on the figure. This means that even in a transient region of the system (near the start of the puck trajectory) the simulation model matches the physical system quite well. It should be noted that much of the difference between the two plots is in the density (or lack thereof) of data points in the physical experiment. Additional data would help to resolve the attractor.

While an interesting result, the comparison begins to

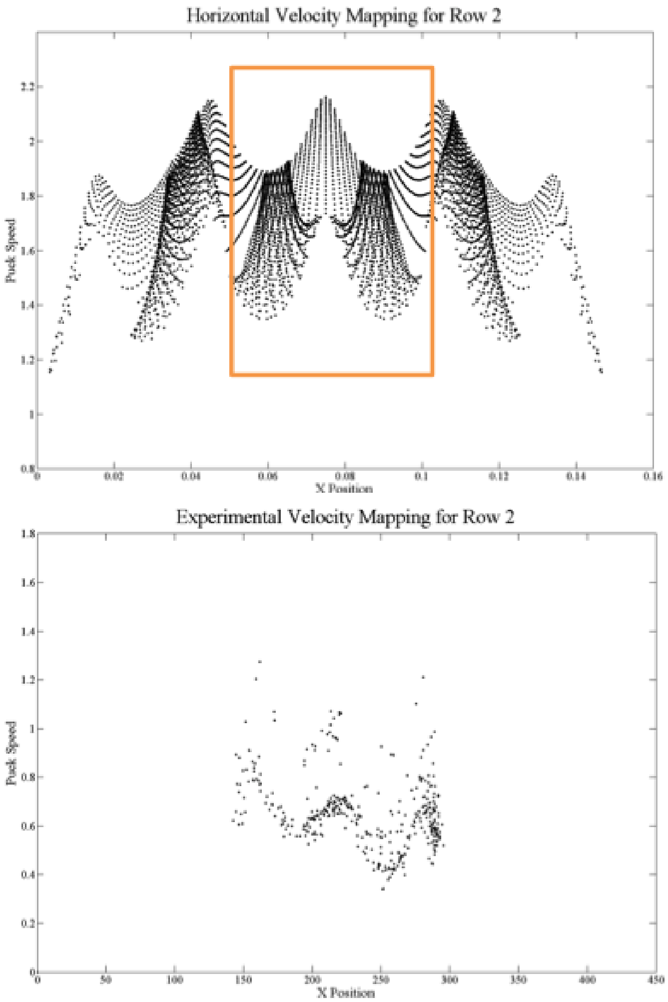

FIG. 9. These figures show the speed magnitude of the puck versus its x location at the second row of magnets from a grid of staring points. The top figure shows the simulation results with an orange box highlighting the approximate area of interest for the experiment. The bottom figure shows the experimental results.

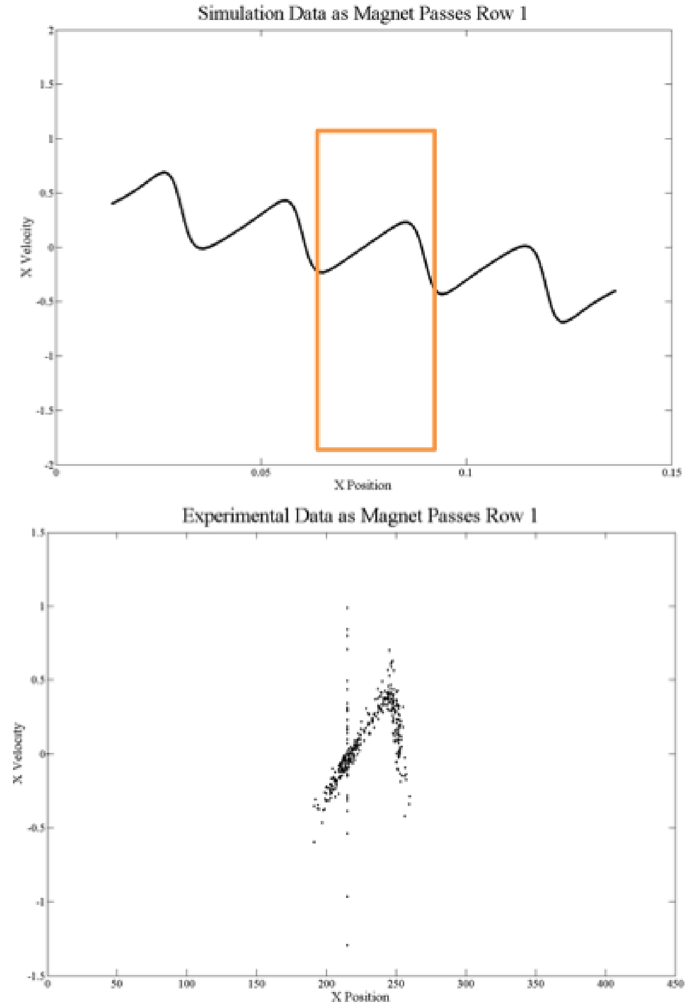

FIG. 10. These figures show the cross-slope velocity of the puck versus its cross-slope location at the first row of magnets from a line of co-altitude of staring points. The top figure shows the simulation results with an orange box highlighting the approximate area of interest for the experiment. The bottom figure shows the experimental results.

break down the further the puck traverses the slope. This is because of the small number of data points in the experiment and the fact that the attractor tends to scatter these points along the map due to their slight differences in initial conditions. What is of interest is that there is a definite order to the evolution of the puck's motion from an ordered, square field of initial conditions to an equally-ordered, folded and contracted phase space after only a short period of time.

Equally interesting and better suited for comparison is the second reduction method. In this method, a line of co-altitude initial points (same y value) was selected that spanned a portion of the cross-slope with various initial x positions included in the set. The cross-slope velocity (magnitude and direction), u, of the puck was then observed as it passed each row of magnets and the initial cross-slope x position was plotted against the velocity at each row of magnets. This is shown as a progression from Row 1 (FIG. 10 to Row 3 (FIG. 11).

This result is far more interesting for several reasons. First, the results of the simulation and experiment remain comparable. The data at Row 1 (FIG. 10) bears this out clearly; the data at Row 3 (FIG. 11), while less immediately clear, can be seen to have a similar qualitative behavior to that predicted by the simulation; there is a difference in the sign of the mean slope of the graphs, owing to the right-biased offset in the experimental data. This can be visually rectified by focusing on the region of the simulated results just left of the vertical center line.

The second reason for the increased interest in this attractor construction method is due to the more obvious folding and contraction of the phase space. It is quite

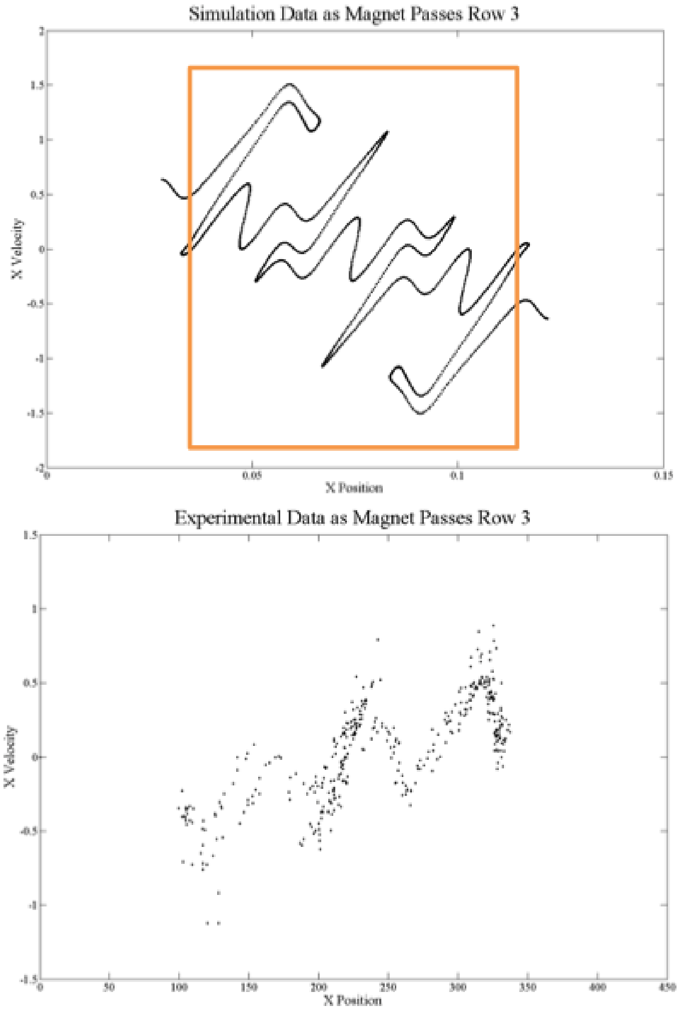

FIG. 11. These figures show how the trajectories in FIG. 10 have evolved by the time they reach the third row of magnets. The top figure is the simulated data, and the bottom figure is the experimental data.

clear at this point that there is a definite order and symmetry to the time evolution of the puck due entirely to the starting position. What is more, it can been shown through visual inspection that there are motions that are essentially disallowed by the physics of the system. A puck starting to the right of its nearest peg will go left, and a puck starting to the left of its nearest peg will go right (exactly as expected for the attracting case). However, as the system evolves, there are breaks in this symmetry. Looking at Row 3 (FIG. 11), there are portions of the phase space where the puck encounters a "decision" in that very slight differences in the starting position will cause it to either maintain its diagonal trajectory, reverse directions, or begin a near-vertical descent (the puck will always have an odd number of possibilities stemming from the choice of going high or low on the current trajectory, high or low on the reverse trajectory, or vertical). At each row of magnets, the number of possibilities increases with $1+2(n-1)$ where n is the current row number.

An analysis of the experimental trajectories shows this observation clearly (FIG. 6). Puck trajectories will tend to be attracted onto paths which take the puck the shortest distance between magnets. This trajectory can be described as a 1:1 trajectory: the puck moves through the field on perfect diagonals, traversing an equal distance in x and y in equal time. The break in this pattern comes when the puck comes sufficiently close to a magnet to divert the trajectory straight down. Once the trajectory is straight down, forces from magnets to the left and right cancel, helping the puck to maintain this trajectory. Of course, because of sensitivity to the initial conditions at each row crossing, this trajectory eventually diverges back to a 1:1 line. Thus, the trajectories repeatedly converge and diverge as the system evolves (see Section IV).

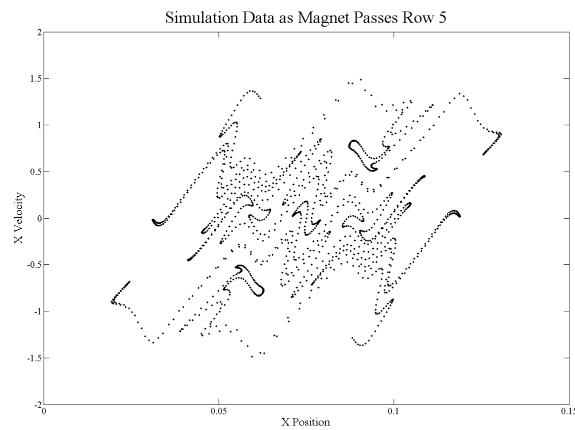

FIG. 12. This figure shows how the trajectories in FIG. 10 have evolved by the time they reach the final row of magnets. Only the simulated results are shown due to the spareness of the experimental data.

At Row 5 (FIG. 10), even the simulated results become difficult to interpret, but it is clear that there is still a structure to the behavior that sees the puck never venturing far from a prescribed motion and making a "decision" about future motion at each magnet row.

Basins of Attraction

Once the simulation was calibrated and validated (to the degree possible) with experimental data, it was possible to use the simulated model to map the basins of attraction for the system at the final row of magnets (FIG. 13 shows the left half of the starting condition space with the right side representing the line of symmetry in the system).

The model shows that there are clear *regions* of attraction in the system, though these regions are not the

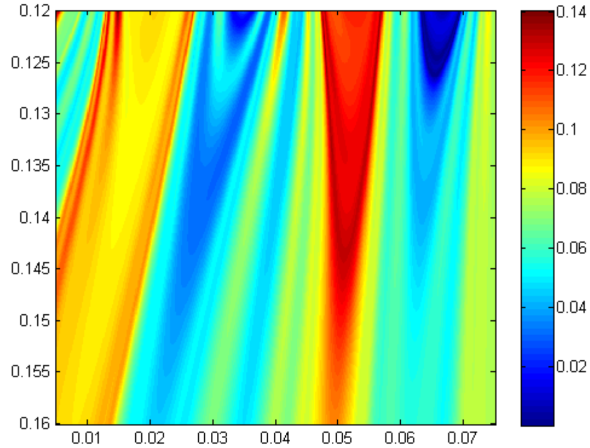

FIG. 13. The basins of attraction for the left half of the slope are shown for y_0 versus x_0. The colors on the plot show the final cross-slope position with red representing the center of the plane and blue the outer edge.

isolated basins typically found in a chaotic system. Instead, these large attracting regions are smeared (likely due to gravity) and line up with the top row of magnets. There are, however, small regions where points diverge from those around them such as on the $y_0 = 0.12$ line around $x_0 = 0.04$. In this region, the trajectories tend toward the $x_f = 0.04$ to $x_f = 0.05$ end state with the lone exception of a small band right at $x_0 = 0.04$ which tend toward an end state nearer $x_f = 0.12$.

This certainly shows the sensitivity of the system to its initial conditions and demonstrates the attractor's effect on the puck during its downward traversal (note that the attractor maps from the previous section correspond to $y_0 = 0.14$), but the hope that this is indicative of self-similarity in the basin map is not supported by the data. A detailed mapping of the region showing the extreme sensitivity did not reveal self-similarity in the map and therefore no Cantor set of the initial position to final position map is expected.

Steady-State Magnetic Ski Slope

The previous results, while interesting, only represent the transient behavior of the system in keeping with the experiment. As the system evolves, the dynamics settle onto the final attractor and become more interesting (FIG. 14). The dynamics show that the trajectories slowly collapse into a column as the system evolves and then suddenly the system expands away from the center line onto the attracting 1:1 trajectory groups. While this figure shows the phase space starting to contract onto these 1:1 trajectories, it is clear the the most interesting dynamics have yet to appear in the system as it has not yet reach a ”steady state.”

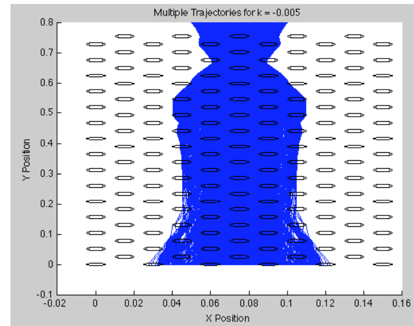

FIG. 14. The system's long-term behavior demonstrates the oscillatory expansion and subsequent contraction of the phase space.

Reducing the number of runs to a small group of similar starting positions shows that the runs remain on a remarkably similar 1:1 trajectory, as previously described, until they turn and begin a vertical descent (FIG. 15). It is at this point that the attractor begins to pull these trajectories to different parts of the phase space, thus showing the sensitivity to initial conditions. This behavior is qualitatively similar to behavior demonstrated by the Lorenz normal-force ski slope (FIG. 16).

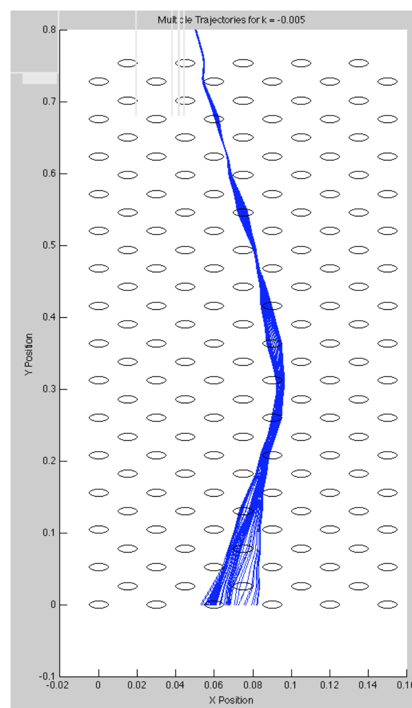

FIG. 15. This subset of trajectories show that similar starting conditions remain close for a long period of time, eventually diverging into chaos.

The behavior of the Lorenz system shows that the dynamics of the normal force ski slope quickly diverge (much quicker than the magnetic ski slope) and provide

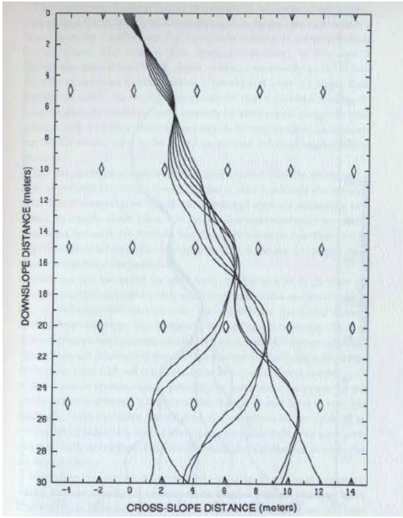

FIG. 16. The Lorenz system diverges into chaos in much the same fashion as is seen in the magnetic ski slope, but much more quickly[4].

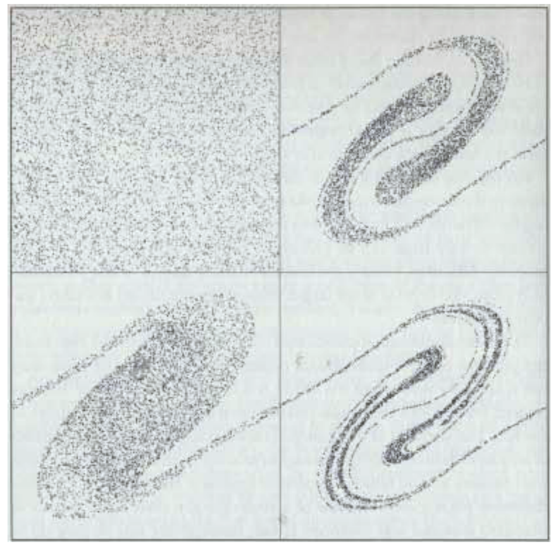

FIG. 17. Evolution of the magnetic attractor from Row 5 to Row 30. The phase space has begun to contract and there is much more order to the attractor structure.

interesting dynamics almost immediately. The magnetic ski slope still has not begun to show the same divergence and full formation of the attractor at the end of this longer simulation. Additional magnetic rows (beyond the 30 shown here) need to be run to further investigate the long term behavior, but the run times are prohibitively long as to have prevented this additional analysis in this study.

In constructing the attractor that forms in the long-term evolution of the system, we hoped to see similar behavior to that of the long-term Lorenz attractor (FIG. 17). This clear spiral was not, however, found (FIG. 18). Instead, the phase space is just showing the beginnings of a collapse into ordered chaos. A longer runtime would have, yet again, provided more insight.

There are, however, a few notable features of the attractor. The 1:1 and vertical attracting trajectories can be easily seen beginning to form globally. The two hooks on the left and right of the attractor represent these bands which are collapsing and expanding trajectories away from the center line. The spirals near the middle of the attractor represent the oscillatory nature of the trajectories near the vertical trajectories at the center line. The inner spiral is the effect of the center line of magnets on trajectories near the center. The outer spirals are the effects of the magnets just surround the center magnet above and below. These help to pull trajectories away from and back towards the center. This clearly ordered system that is beginning to form demonstrates the Cantor-like nature of the trajectories to repeatedly go between vertical and diagonal lines (the paths of least resistance in the Lorenz ski slope and the paths of highest magnetic attraction in the magnetic ski slope).

The dissimilarities between the Lorenz system and the magnetic system are likely due to three main contributions. The first is obvious: the Lorenz system is a passive force system whereas our magnetic system produces active attracting force on the puck at all times and from all pegs. That said, the forces of pegs not near the puck are nearly zero (thus why the finite span of magnets is an acceptable configuration and also likely the cause of the 1:1 trajectory behavior) and aid in keeping the overall nature of our system similar to the Lorenz system. The second significant difference is the set of simplifying assumptions. In the Lorenz study, the downhill speed of the puck is fixed; in this system, it is not fixed and depends on gravity and magnetic attraction which speeds and slows the puck. The Lorenz system is therefore semi-conservative as energy lost or gained from the slope is matched by the puck speeding up or slowing. The final difference is that the Lorenz system was integrated over a much longer slope, allowing the system to fully evolve. Due to computational and schedule limitations, this was not possible for our system.

The final item of interest is the effect of the longer run slope on the basins of attraction. While the initial conditions are all located on a co-altitude line, the mapping still shows interesting structure (FIG. 19). There is a hint of self-similar structures throughout the map, particularly near the $x = 0.06$ line. Unfortunately, the resolution only allows for limited investigation of this structure and shows only a hint of possible self-similarity at this location (FIG. 20).

V. CONCLUSION

The magnetic ski slope shows many of the same qualitative features as the Lorenz normal force ski slope. Both systems exhibit clear attracting behavior with phase space structure (e.g., folding, contraction, dense trajec-

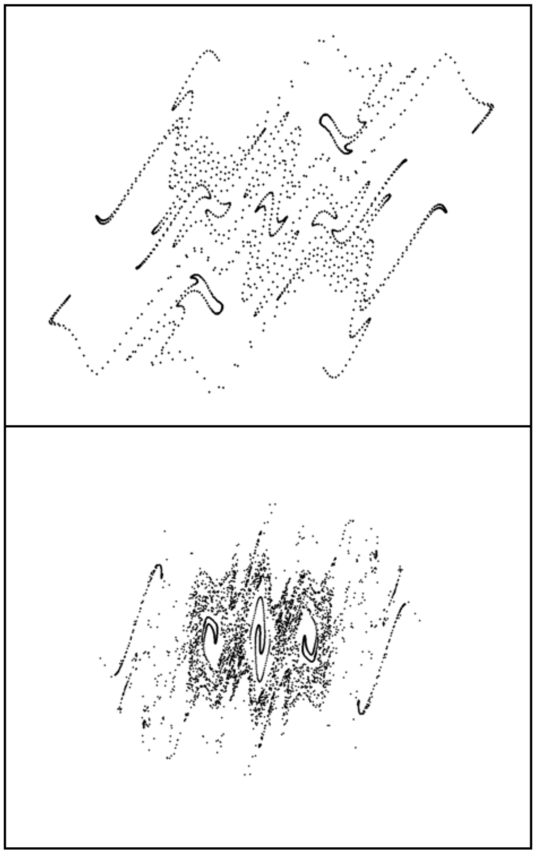

FIG. 18. Lorenz's attractor as generated by his ski slope experiment[4].

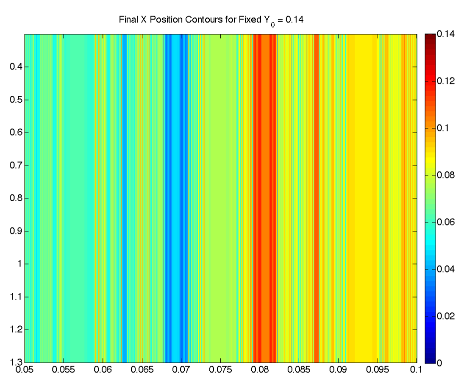

FIG. 19. The basins of attraction are starting to show more sensitivity and possible self-similarity, as one would expect with a chaotic system.

tory regions, structure formation), Cantor-like trajectories, and basins of attraction. This shows that the magnetic system has rich chaotic dynamics that are qualitatively similar to the pedagogical case despite differences in the underlying assumptions (i.e., the "ski slope" problem is somewhat general). Additionally, experimental results, which were not collected by Lorenz, show that

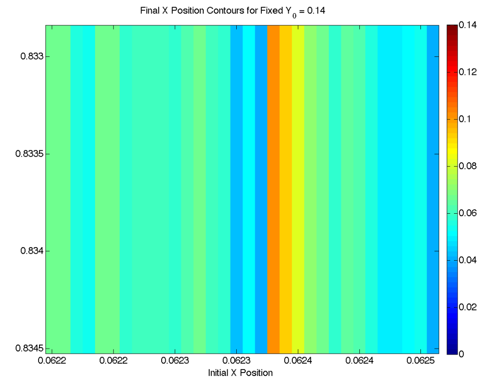

FIG. 20. The region near $x = 0.06$ seems to show self-similarity with the larger map, though the granularity of the image prohibits further investigation.

the qualitative transient behavior of the model matches that of the real system when calibrated with the real experimental conditions (a result which seems to validate the model). The experiment, while limited in slope size, even shows the beginnings of an attractor forming.

The principal shortcomings of the investigation are with the amount of data collected in the experiment, not the quality of the data. Many more runs would be necessary (on both a co-altitude line and in a randomized starting region) on a far longer slope to determine the true, long-term nature of the attractor in the real system. In addition to this, a longer simulated slope would be necessary to fully reconstruct the 3D attractor of the magnetic ski slope in simulation (for comparison with Lorenz) and allow for investigation of self-similarity in the attraction basins map. Future work should also focus on other real-world analogs to this system wherein a magnetized object passes through a lattice of magnets as it rides a continuous-force current (examples that comes to mind are electron motion in a wire with current or charged partical flows in plasma) in order to investigate further the generality of the lattice motion phenomena observed here.

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