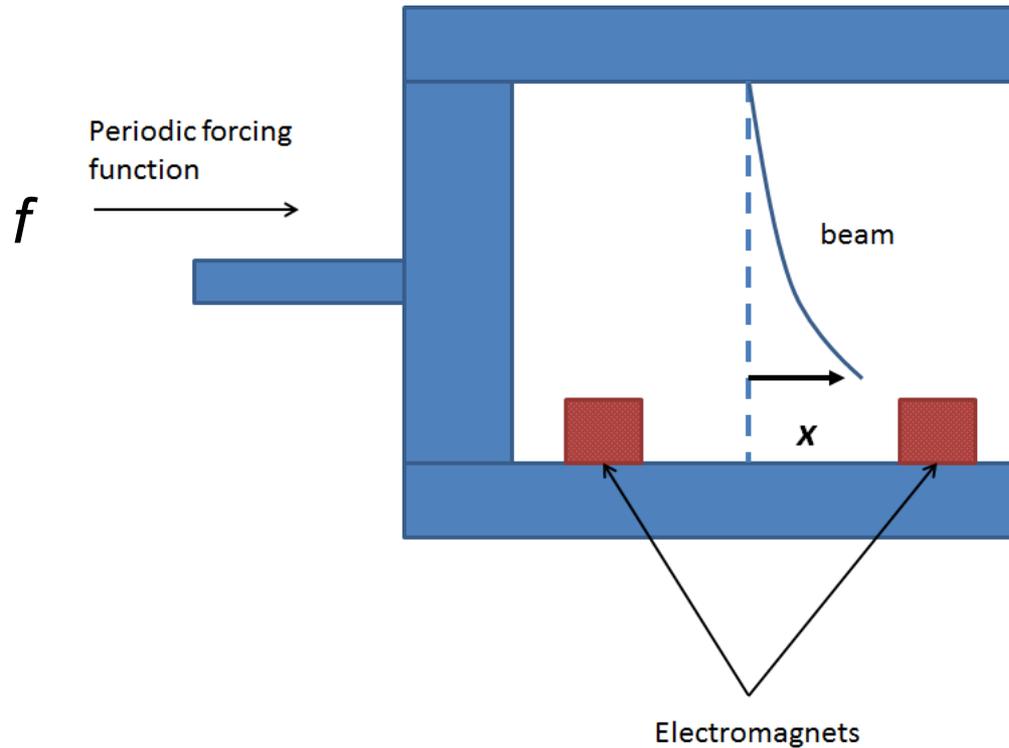


PHYSICS 6268 Project
Dynamics of the Periodically-
Forced Duffing Oscillator

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Representation of the experiment



Introduction

The periodically-forced Duffing oscillator is modeled by the following non-autonomous ODE:

$$x'' + \delta x' - \alpha x + \beta x^3 = f p(\omega t)$$

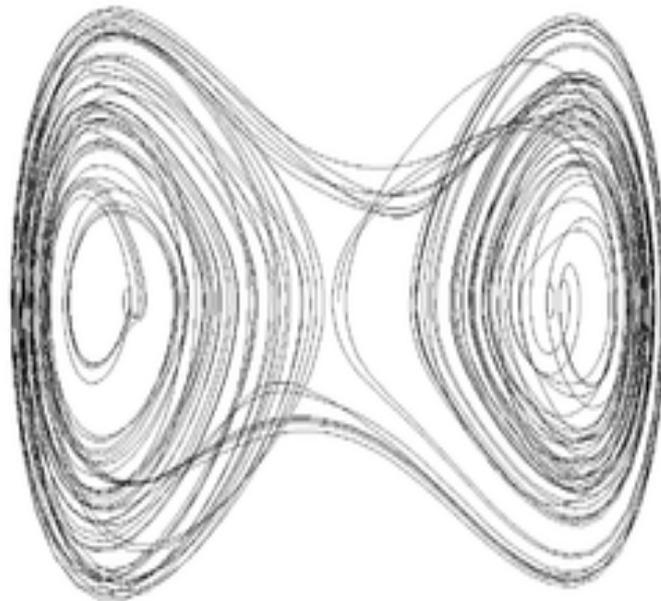
where p is 1-periodic and $\alpha, \beta, \delta, \omega, f > 0$ are constants.

Our goal

- Analyze the bifurcations which occurred for fixed α, β, δ as f (the amplitude of the forcing) and ω (the frequency of the forcing) were increased

Example Phase Portrait

- Without forcing the system has three equilibria
 - Two stable equilibria on the extreme ends
 - Unstable equilibrium in the center



The equation

$$x'' + \delta x' - \alpha x + \beta x^3 = f \cos(\omega t)$$

Case 1: Set $\delta = f = 0$. In this case our equation reduces to

$$x'' - \alpha x + \beta x^3 = 0$$

Newtonian system with potential $V(x) = (\beta/4)x^4 - (\alpha/2)x^2$ and the trajectories are curves of constant energy:

$$E(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + V(x).$$

Using the standard linearization technique, we see that we have a saddle at the origin and centers at $(\pm \sqrt{\alpha/\beta}, 0)$

Other than the two homoclinic orbits from the origin to itself and the two centers, we have that all other trajectories are in fact closed orbits.

The equation

$$x'' + \delta x' - \alpha x + \beta x^3 = f_p(\omega t)$$

Case 2: Include the dissipation term (i.e. we let $\delta > 0$), the system is again easy to understand: heuristically, trajectories spiral into what are now stable fixed points at $(\pm \sqrt{\frac{\alpha}{\beta}}, 0)$. We can see that in this case

$$\frac{d}{dt} E = \dot{x}(\ddot{x} + \beta x^3 - \alpha x) = -\delta \dot{x}^2 \leq 0$$

so the energy decreases (strictly, when $x' \neq 0$) along trajectories of the system. The origin is still a saddle in this case. It can be shown that, except for the stable manifolds of the saddle, all trajectories of this system tend toward one of the stable fixed points.

Transition to Chaos

The orbits become much more complicated for $f > 0$.

We consider the following cases:

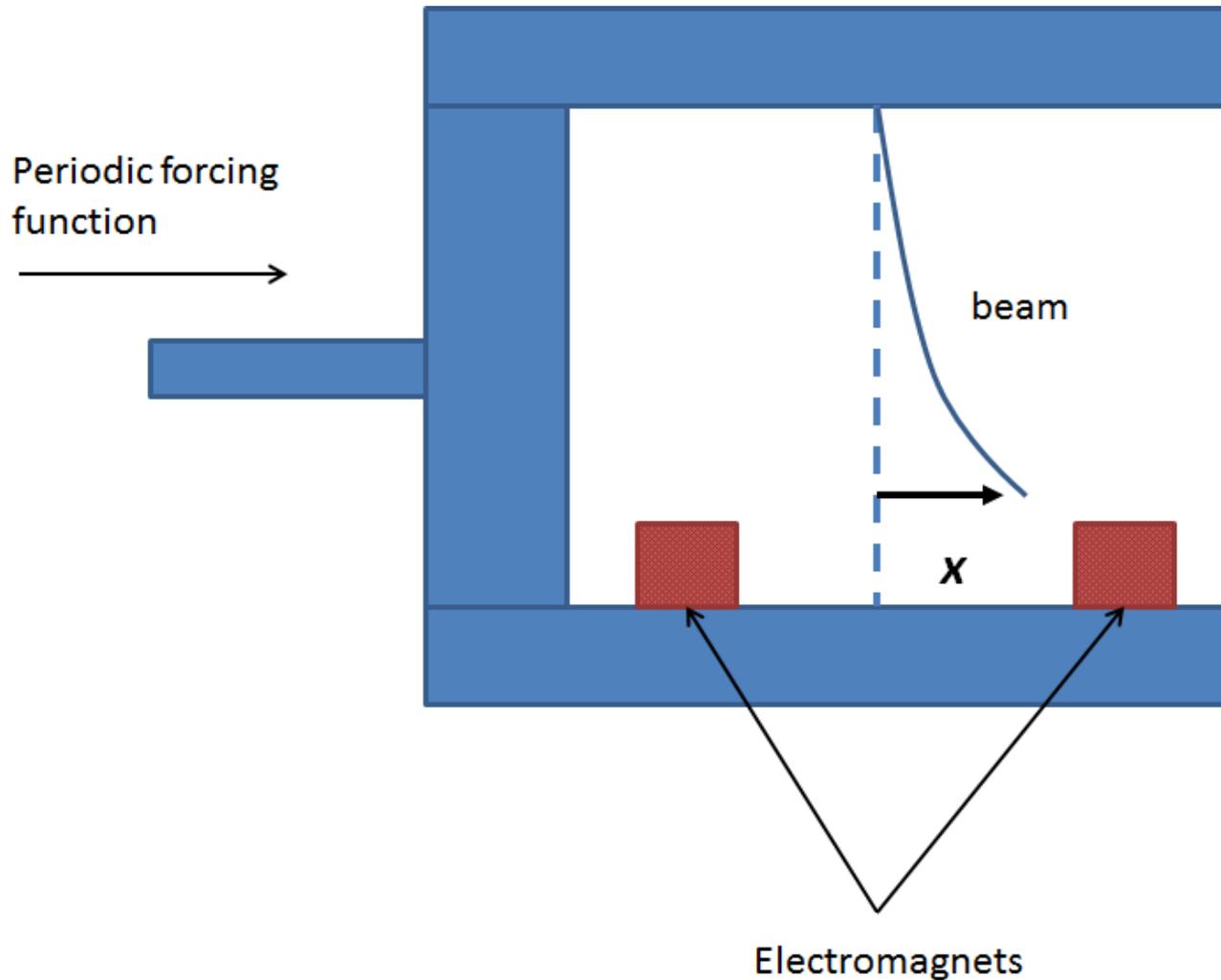
- **Small f :** The behavior is well-understood in the case where p is sinusoidal. In this case, the system closely resembles the non-forced system: the stable fixed points become attracting orbits of period $2\pi\omega$ and we still have something like a saddle near the origin.
- **Large f :** The behavior is also well-understood, since in this case the forcing term ``dominates'' the other terms.

Transition to chaos

- In between these two extremes, the system becomes extremely complicated. In order to understand the behavior of the system with a "mid-range" non-zero forcing term, $f > 0$, it's easier to study and analyze the Poincaré section of the trajectories.

Experimental Setup

Planned Apparatus



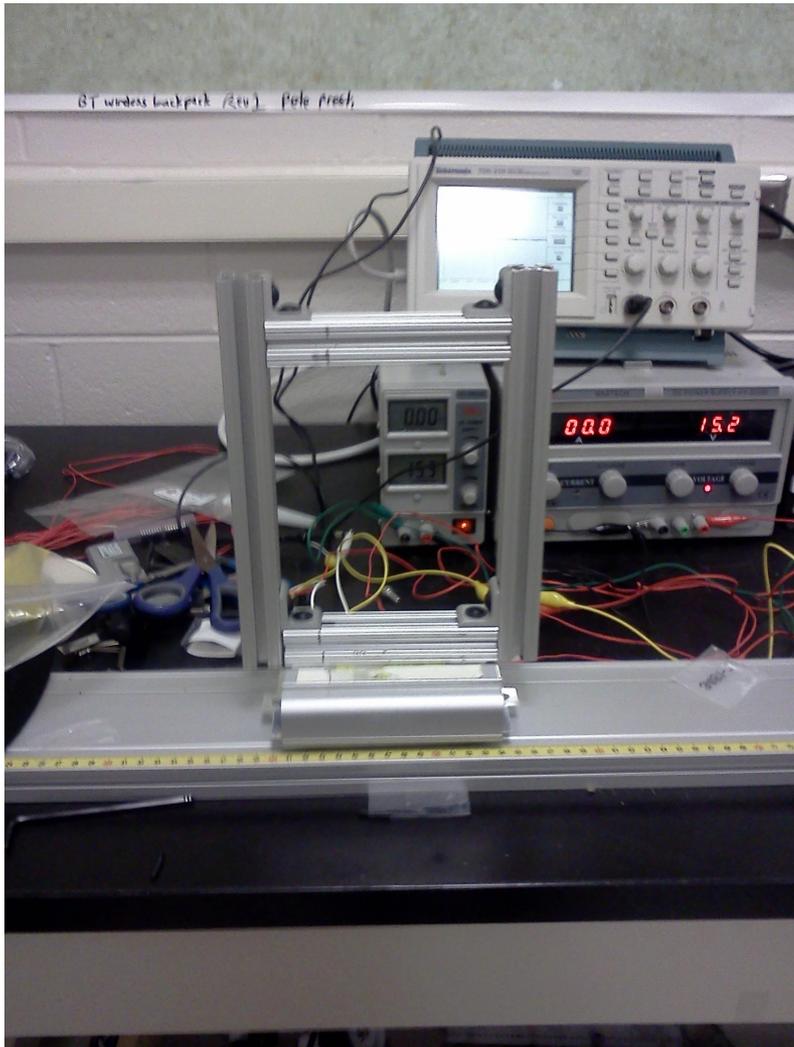
Apparatus Construction

- Beam and magnet selection
 - Beam cut from shim stock
 - Attaching magnets or weights to end of beam proved difficult
 - Without attachments at end of beam, needed powerful stationary magnets
 - Used rare-earth magnets instead of electromagnets
 - Well potentials adjusted through magnet position rather than electrical control
- Sensors
 - Strain gauges attached near base of beam
 - Amplified using voltage differential circuit
 - Good accuracy and precision, minimal drift, adequate range
 - Camera not needed



Shim Straightening

Apparatus Construction: Locomotion and Chassis



- Wheeled cart on track
- 80/20 frame
- Stationary motor; motor arm attached to cart

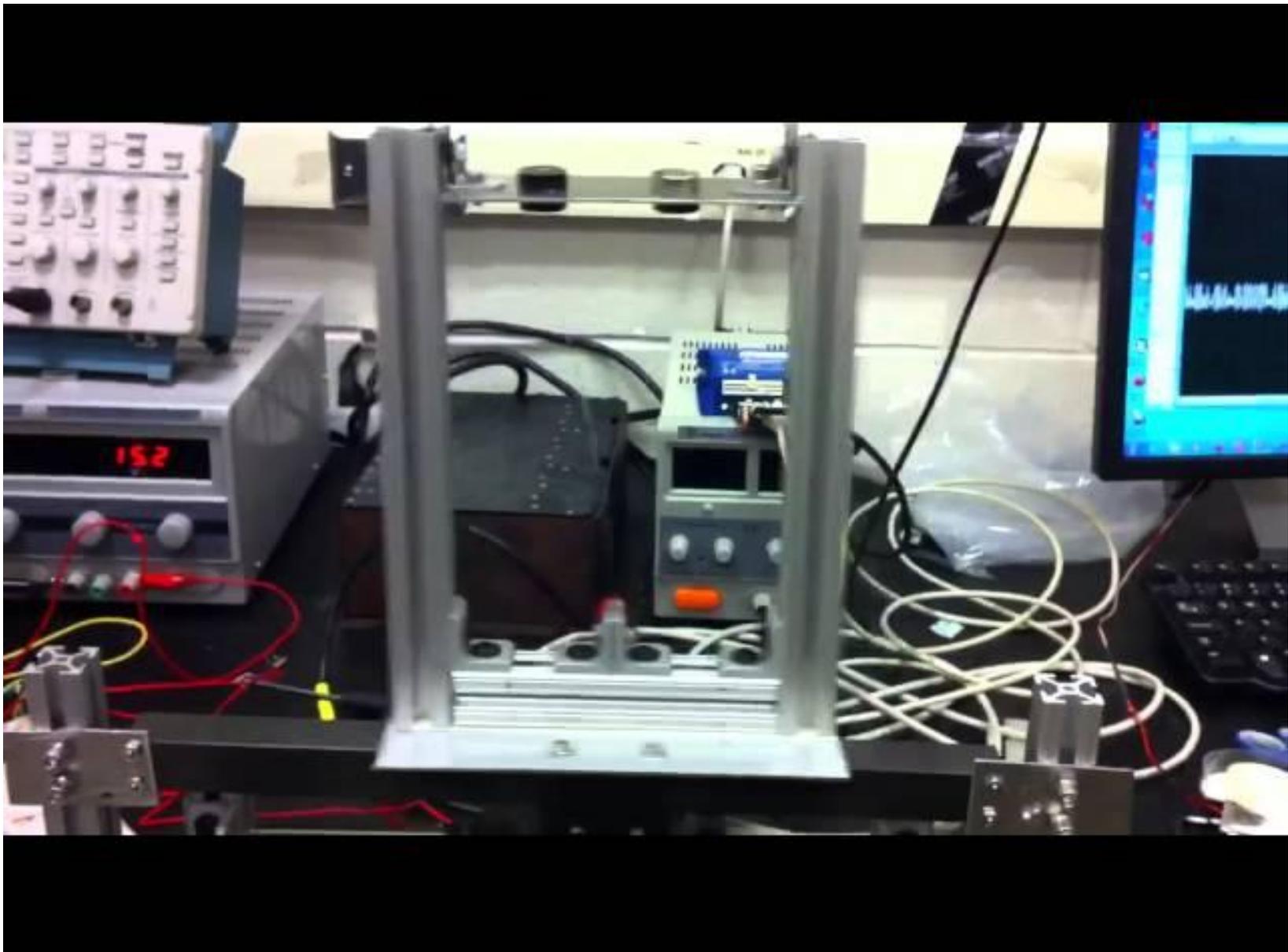
Problems:

- High friction, wheel binding
- Difficult motor attachment

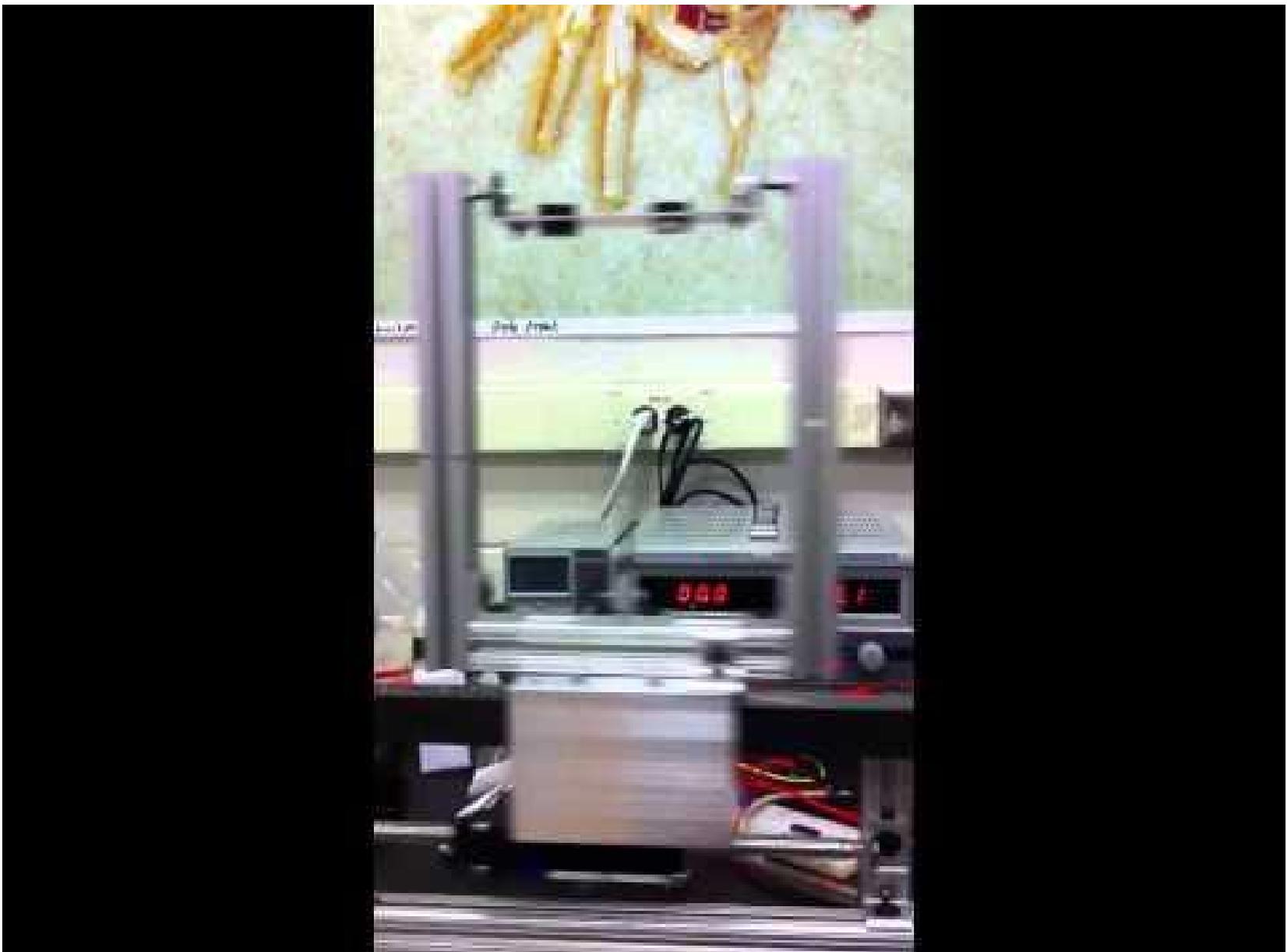
Apparatus Construction: Locomotion and Chassis

- Replaced cart & track with air track
- Motor attached to bottom of track





Video of Apparatus in Operation



Video of Beam Becoming Trapped in Well

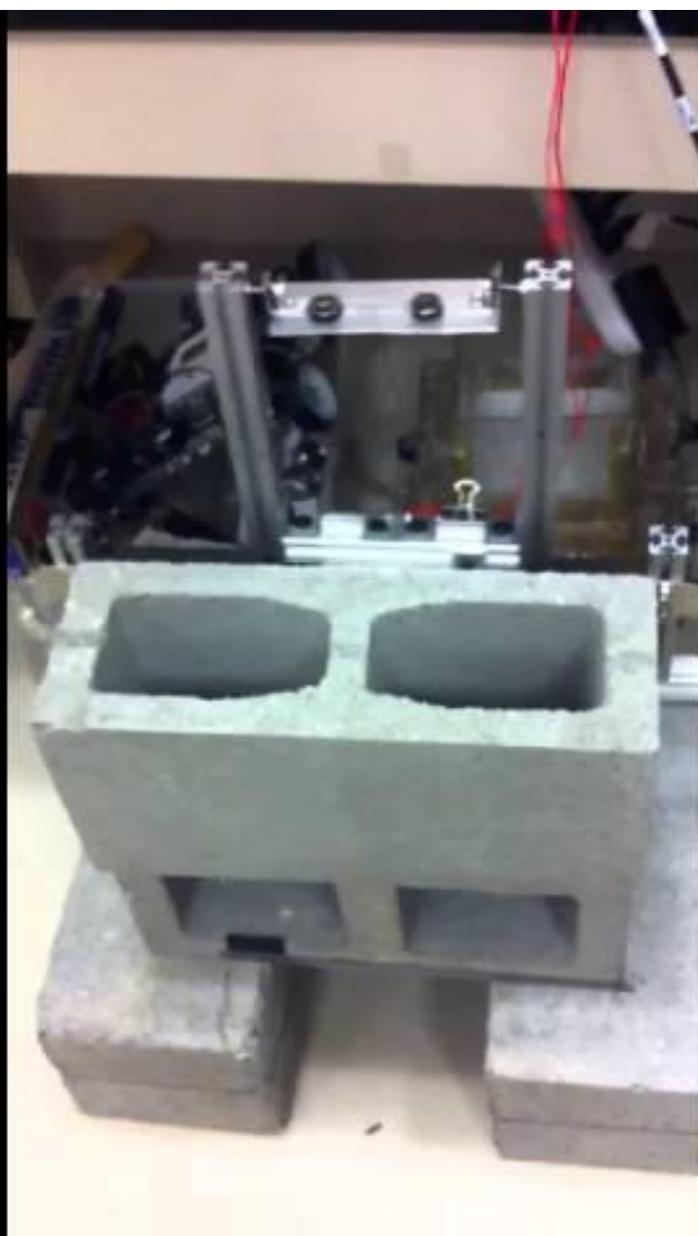
Apparatus Construction

Problems:

- Beam could strike frame during chaotic motion
- Vibration of table and entire apparatus

Solutions:

- Widen frame
- Mount apparatus on cinderblocks adhered to floor



Video of Latest Iteration of Apparatus

Current Issues

- Heat dissipation in motor and CAN amp
- Current limitations of motor
- Asymmetry due to difference in magnet strength and the natural bend of the stock

Software and Automation

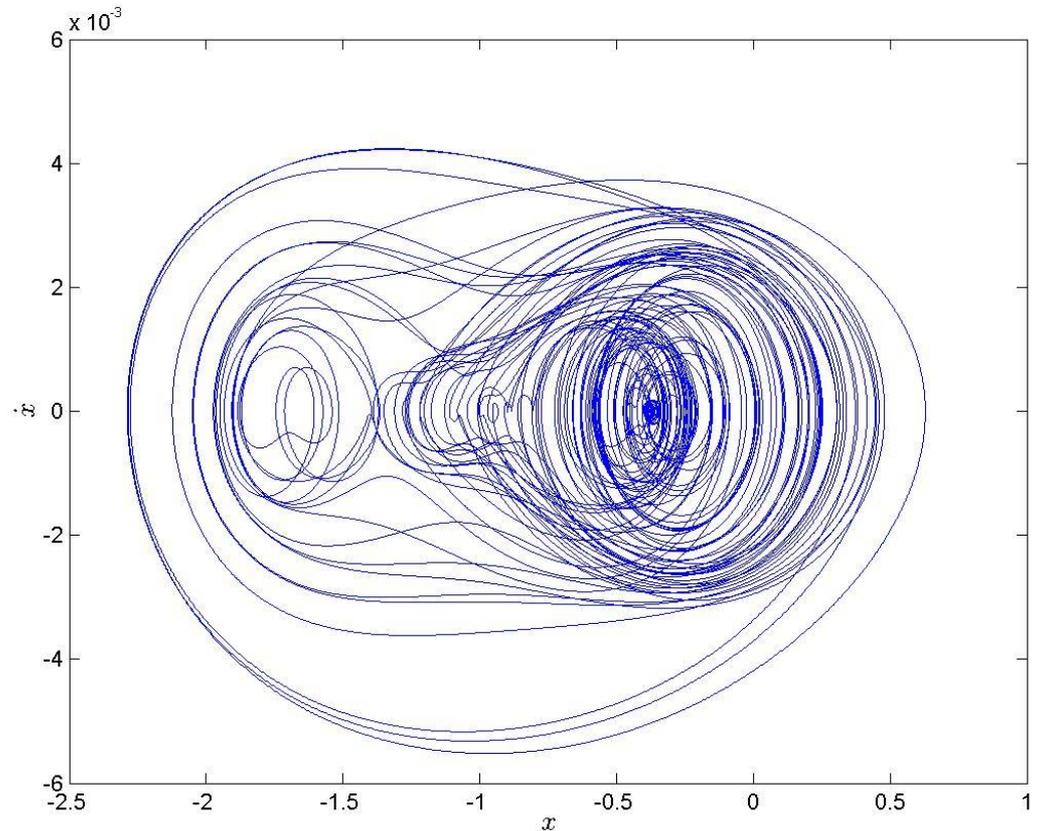
- Copley CAN amp software did not allow non-integer sine frequency forcing, so adapted motor control software from Jeff Aguilar
- Extended to allow parameter sweeps on a grid of amplitude and frequency on definable intervals, step size, and trial repetitions
- Resets beam to left well initial condition through stochastic jerk
- Detects early termination of trial if the beam becomes stuck in a well
- Monitors amp temperature and allows cooling periods for amp and motor
- Records strain gauge voltage measurements for each trial

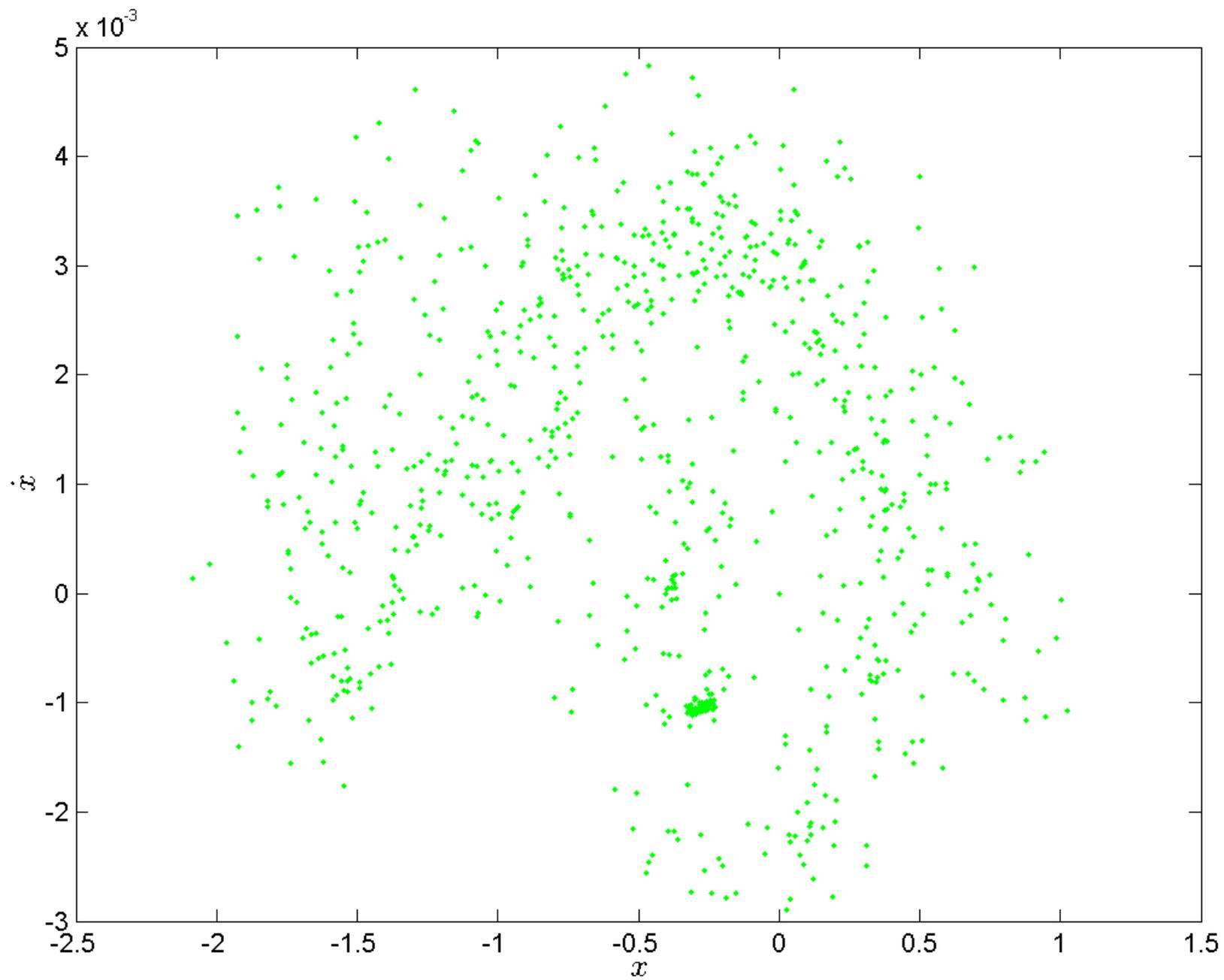
Since the position of the motor is given by $A\sin(\omega t)$, the max acceleration is $A\omega^2$. We found the boundary between chaotic and non-chaotic behavior to be approximately given by a curve $A\omega^2 = \text{const}$. When automating the apparatus, we used curves of this form to define the regions of parameter space we wanted to explore.

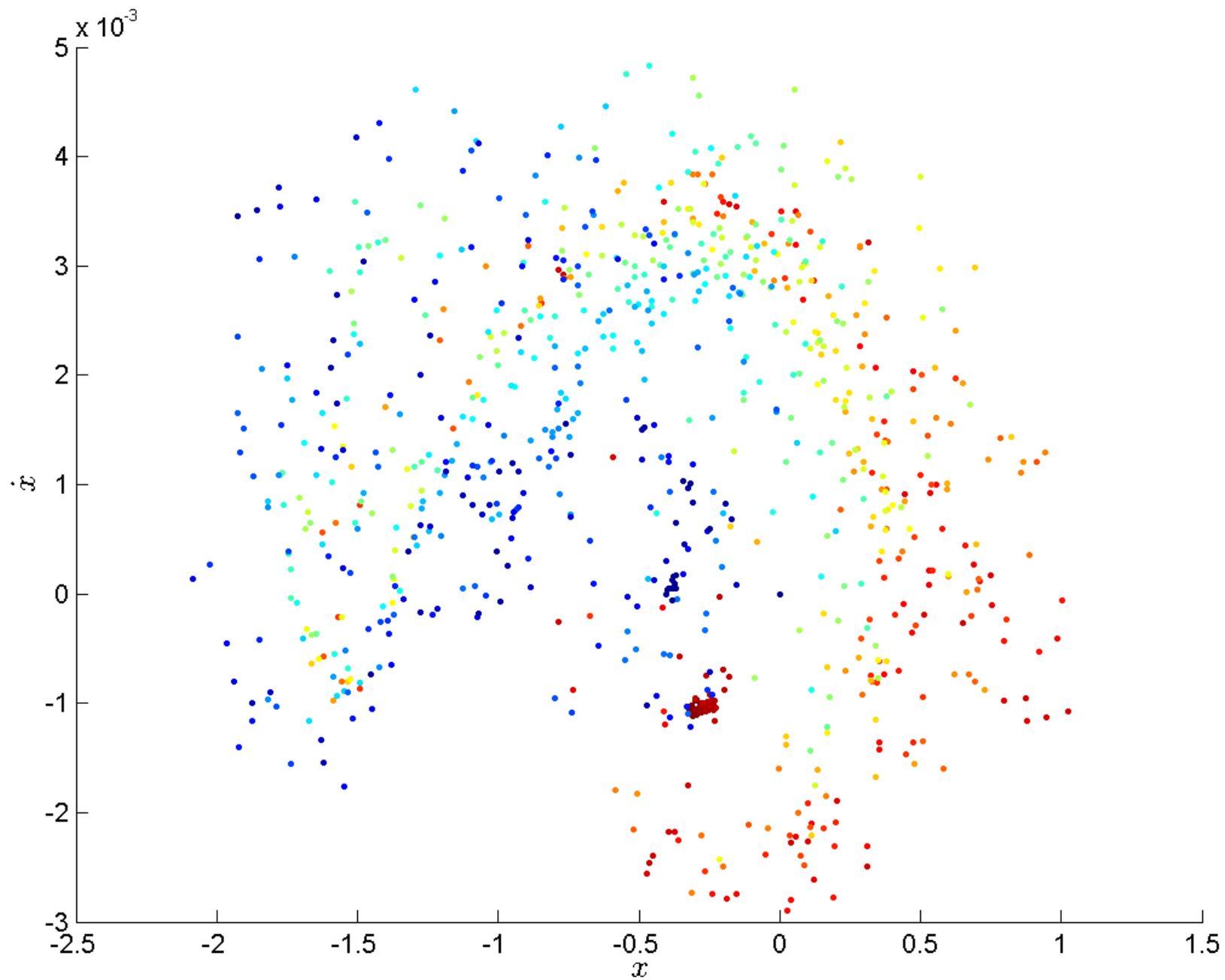
Results

Analysis of One Chaotic Trial

- Sinusoidal forcing
- 2100 counts amplitude
- 4Hz frequency
- Duration: 200 seconds

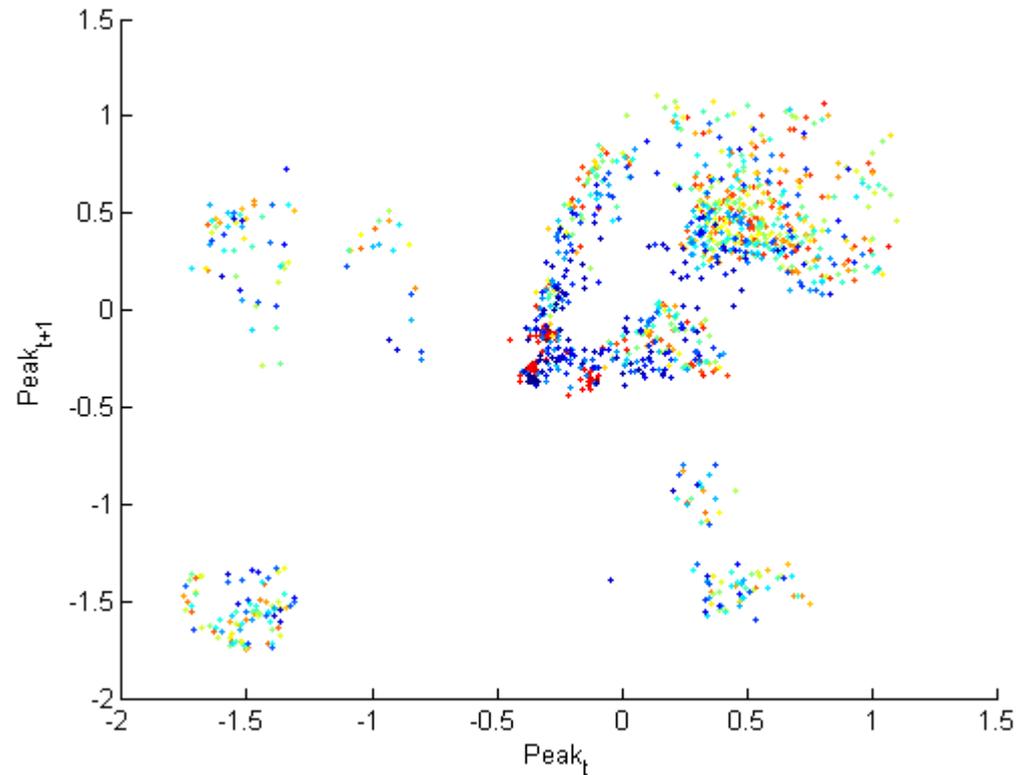






Map of Successive Peaks

Because x is positive while near one well and negative while near the other, four regions form in map, two for remaining in each well and two for transitioning between wells.



Videos of Attractor Reconstruction

Problems, sources of error

-how strong is the correlation between the strain gauge reading and the position of the end of the beam: the strain gauge is measuring "local deflection" - do we still get an accurate representation of the position of the tip of the stock?

-what are δ, α, β, f in our experiment?

-discrepancy between "ideal p" and "actual p"

Future directions in experiment, simulation, and theory

Explore dynamics with non-sinusoidal forcing:

What if p is a trigonometric polynomial?

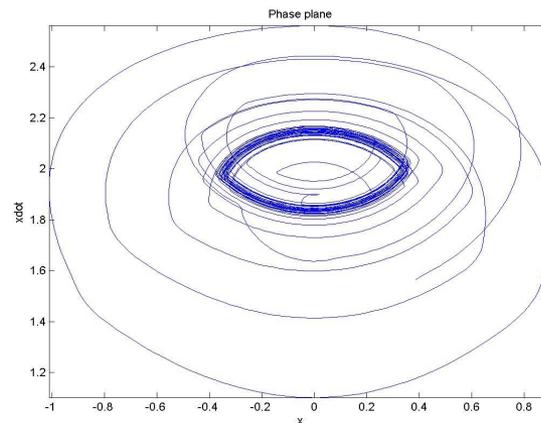
What is known when p is non-smooth?

What if p is, say, $C^{0,\alpha}(\Omega)$?

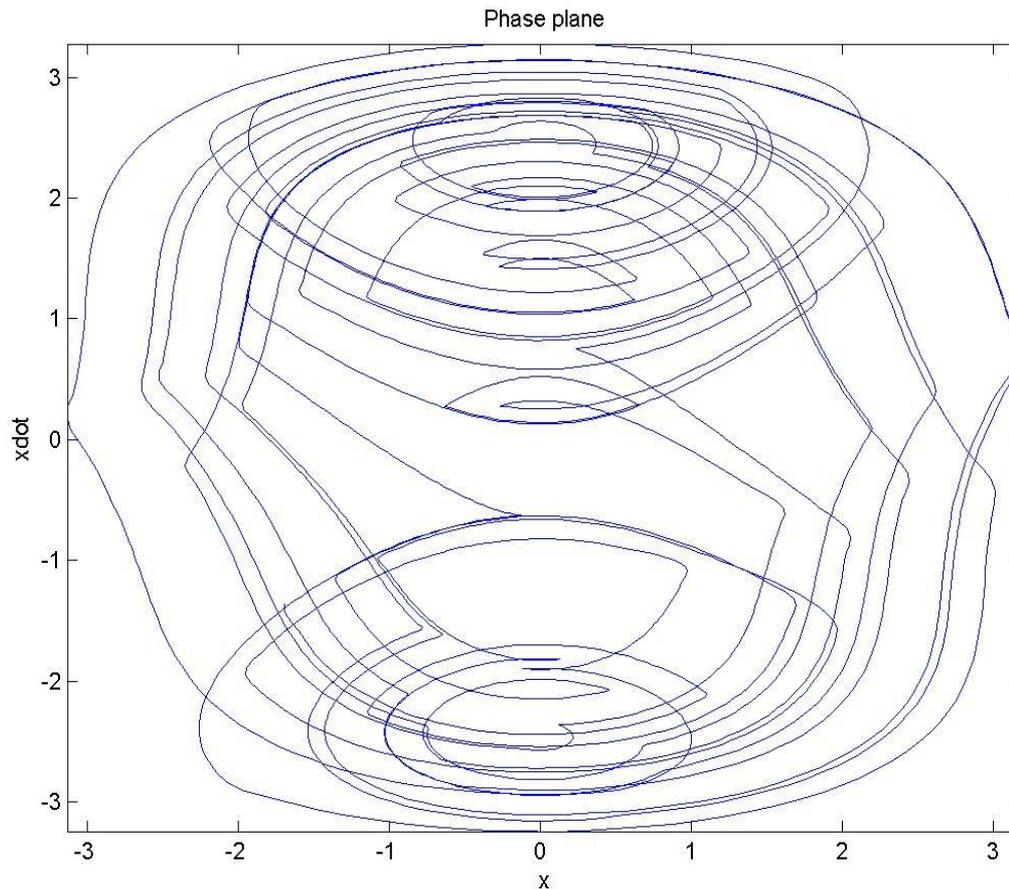
What if p has discontinuities?

Matlab Simulation with Square Wave Forcing

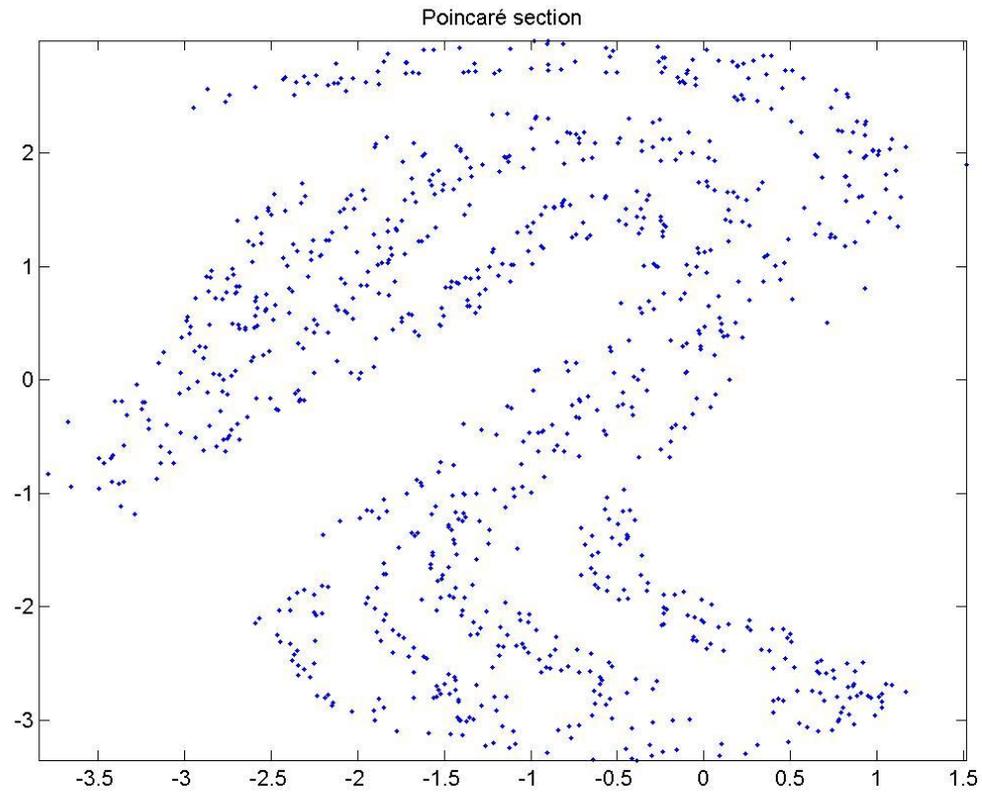
For small f , we observe the existence of limit cycles. Intuitively, the beam is shaking near one of the magnets, but not with enough force to hop to the other magnet. Below we have $\delta=.1$, $\alpha=1$, $\beta=.25$, $\omega= 2$, and $f=.25$ and p is a square wave.



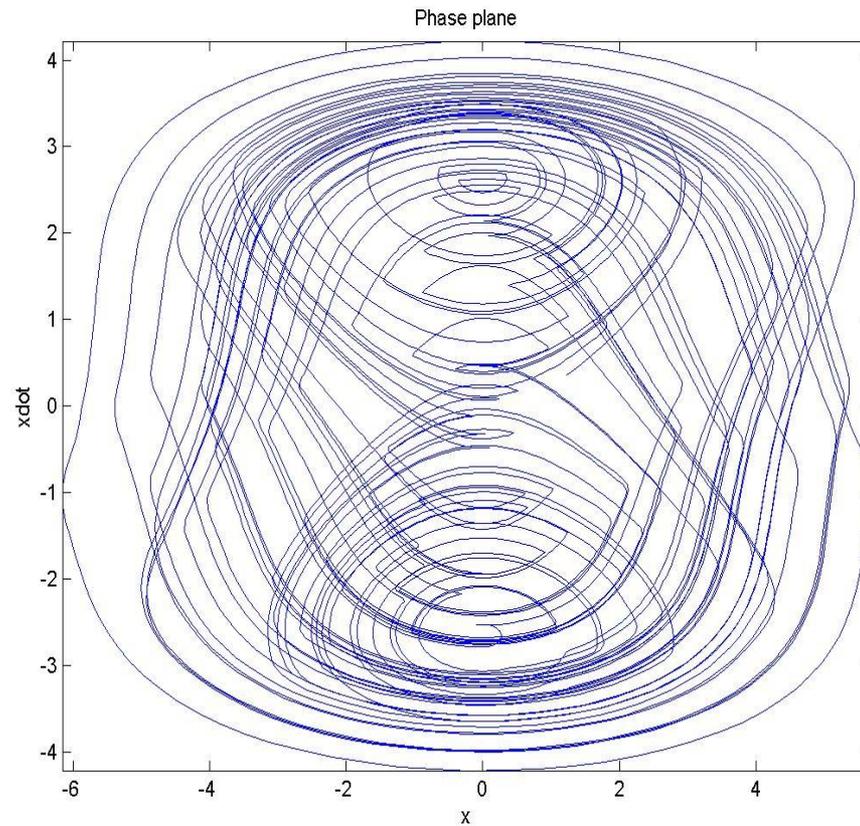
$\delta=.1$, $\alpha=1$, $\beta=.25$, $\omega=2$, p is a square wave.
Now set $f=1.25$



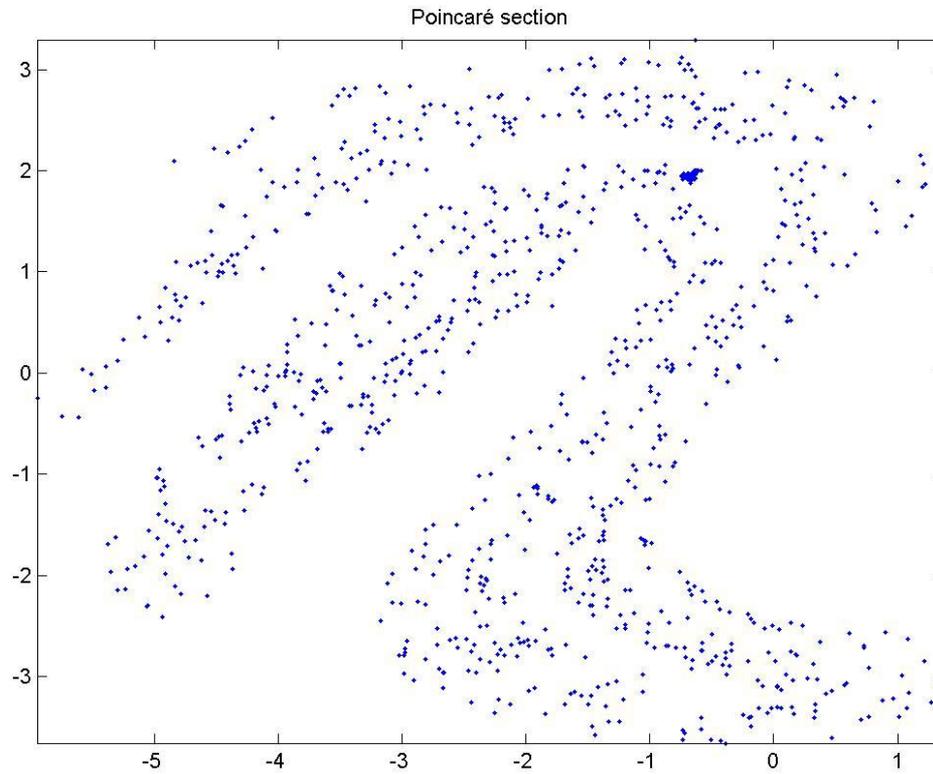
Poincare Section ($f=1.25$)



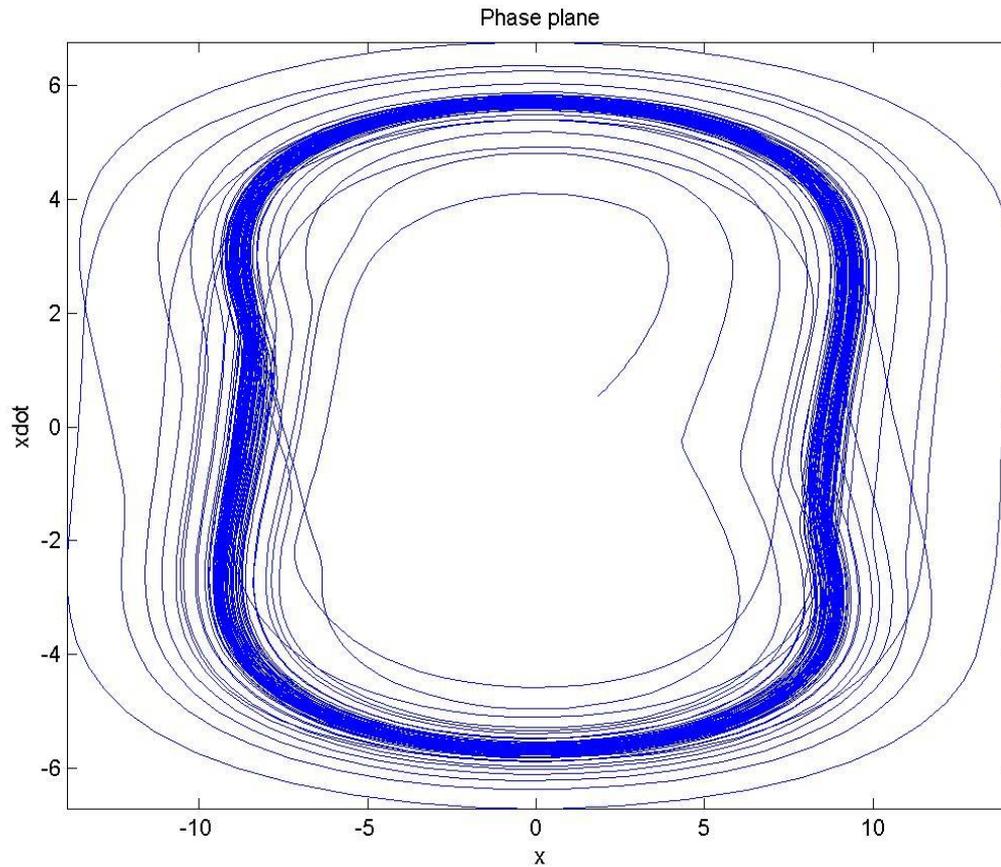
$\delta=.1$, $\alpha=1$, $\beta=.25$, $\omega=2$, p is a square wave.
Now set $f=2.0$



Poincaré Section ($f=2.0$)



$\delta=.1$, $\alpha=1$, $\beta=.25$, $\omega=2$, p is a square wave.
Now set $f=3.0$



Problem (perhaps easy)

Analytically demonstrate the existence of a limit cycle for sufficiently small and sufficiently large f in the case of square wave forcing.

What about when p is (say) any nonzero bounded periodic function with integral 0?

If so what is the weakest regularity you can assume p has?

A nice java simulation of the sinusoidally forced duffing oscillator:

<http://www.math.udel.edu/~hsiao/m302/JavaTools/osduffng.html>