The Inverted Pendulum

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The inverted pendulum is archetypal to both Control Theory[1, 2] and Nonlinear Dynamics[3].

[1] D. Liberzon. Switching in Systems and Control (2003 Springer)
[2] Franklin; et al. (2005). Feedback control of dynamic systems, 5, Prentice Hall.
[3] Strogatz, Steven (1994). Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering. Perseus Books.

History of a persistent problem

First investigated by Kapitza, then H.P. Kalmus, and E.D. Yorke.

Kapitza is a Nobel laurate and founded The Institute for Physical Problems at the Russian Academy of Sciences.

E.D. Yorke is one-half the Yorke-named team behind the Kaplan-Yorke number for the dimension of the strange attractor.

the striking and instructive phenomenon of dynamical stability of the turned pendulum not only entered no contemporary handbook on mechanics but is also nearly unknown to the wide circle of specialists…
…not less striking than the spinning top and as instructive.

P.L. Kapitza

Collected papers of P. L. Kapitza, Vol. 2, edited by D. Ter Haar (Pergamon, London, 1965). p. 714, 726

Two Inverted Pendulum Problems

Vertically Driven Base

If the driving waveform is known *a priori*, the system reduces to a Lagrangian of a single variable.

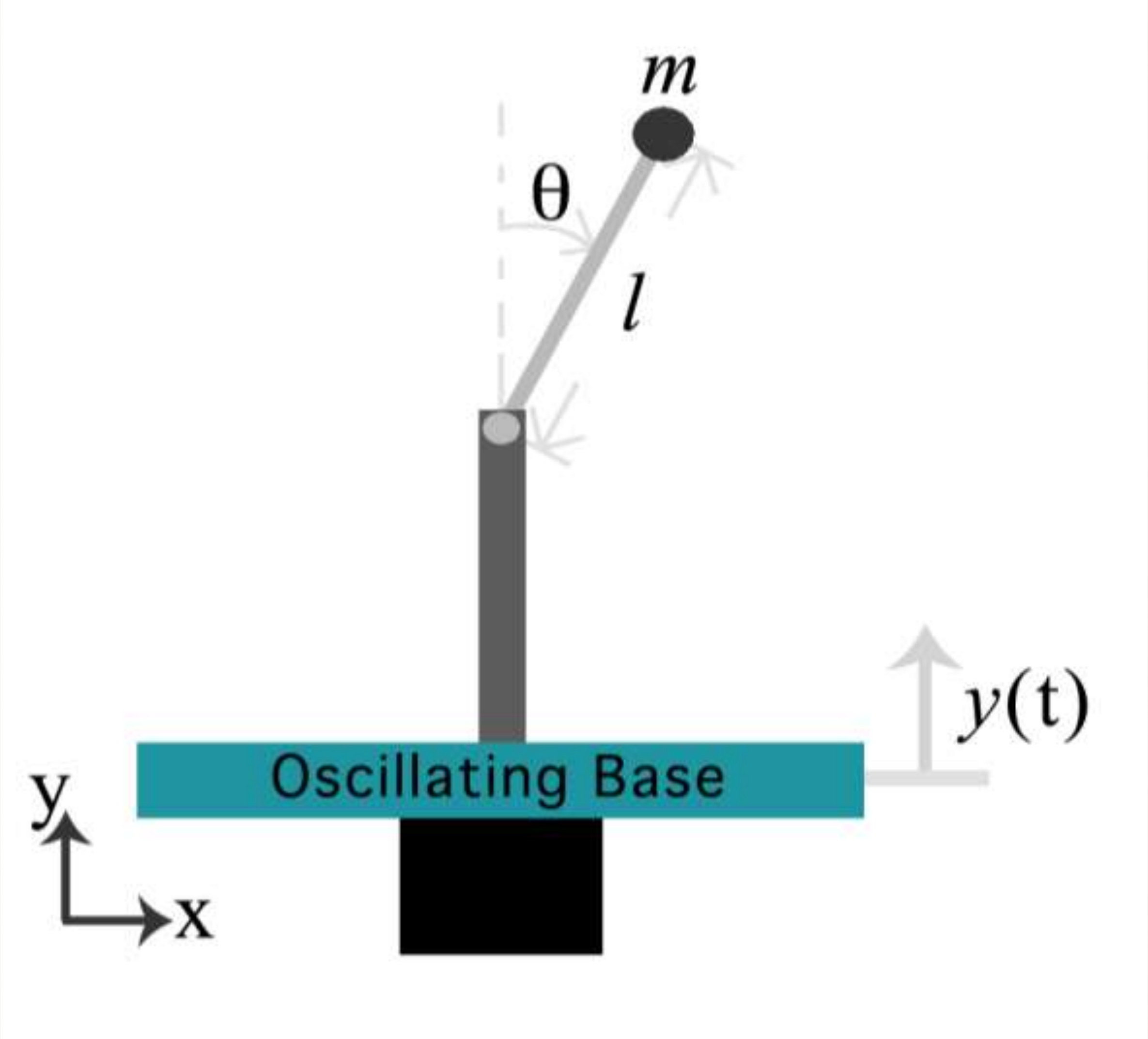
Sinusoidal waveforms are best waveforms.

Vertically Constrained

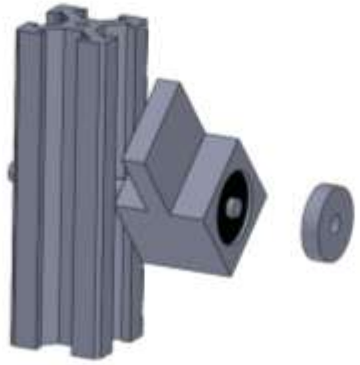
Typically involves a freely moving cart and a track; irreducibly two variable.

More of a Control Theory problem, really.

The
Vertically Driven Inverted Pendulum



Theory and Circumstance



Single Pendulum

- A simple pendulum consisting of a solid piece of mass m.
- It has two equilibrium configurations: the *stable down* and the *unstable up* position.
- By applying a harmonic vertical displacement $(y = A\sin(\omega t))$ at the pivot, the inverted state of the pendulum can become stable within a bounded range of amplitudes and frequencies.

Derivation

Lagrangian of the inverted pendulum with a vertically-driven pivot:

$$\mathcal{L} = \frac{m}{2}\left(l^2\dot{\theta}^2 + \dot{y}^2 + 2l\dot{y}\dot{\theta}\sin\theta\right) - mg(y(t) + l\cos\theta)$$

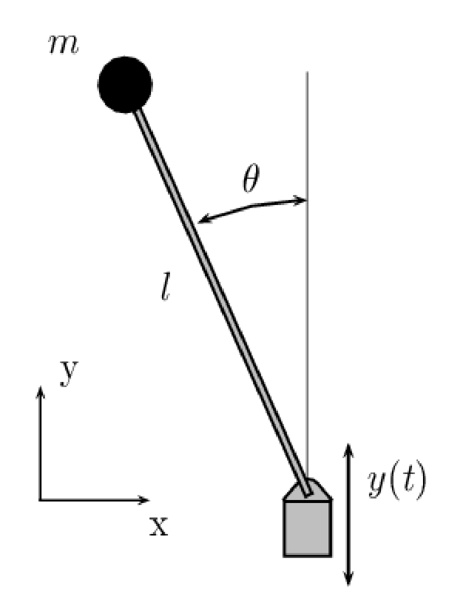
θ– angle between the pendulum arm and upward vertical in a counterclockwise direction

$\dot{\theta}$– first derivative of θ with respect to t

l– length of pendulum

m – mass of pendulum

g – acceleration due to gravity



Rescaling

Solving the first-order Lagrange-Euler equation in $\dot{\theta}$ and θ:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

And re-scaling yields:

$$\ddot{\theta} + (\beta f(\tau) - \alpha)\sin\theta = 0$$

where, $\ddot{\theta}$ – second derivative of θ with respect to non-dimensional time: $\tau = \omega t$

ω– driving frequency of pivot

$$\alpha = \frac{g}{l\omega^2}$$

g– acceleration due to gravity

$$\beta = \frac{b}{l}$$

$f(\tau)$ – normalized driving function, such that $\partial_\tau{}^2 y(t) = bf(\tau)$

Damping

With damping, the second-order system can be written:

$$\ddot{\theta} + \gamma\dot{\theta} + (\beta f(\tau) - \alpha)\sin\theta = 0$$

γ – constant, scaled friction term

Linearization

The fixed points are:

$$\left(\theta^*, \dot{\theta}^*\right)_+ = (0,0)$$
$$\left(\theta^*, \dot{\theta}^*\right)_- = (\pi, 0)$$

Making the local transformation:

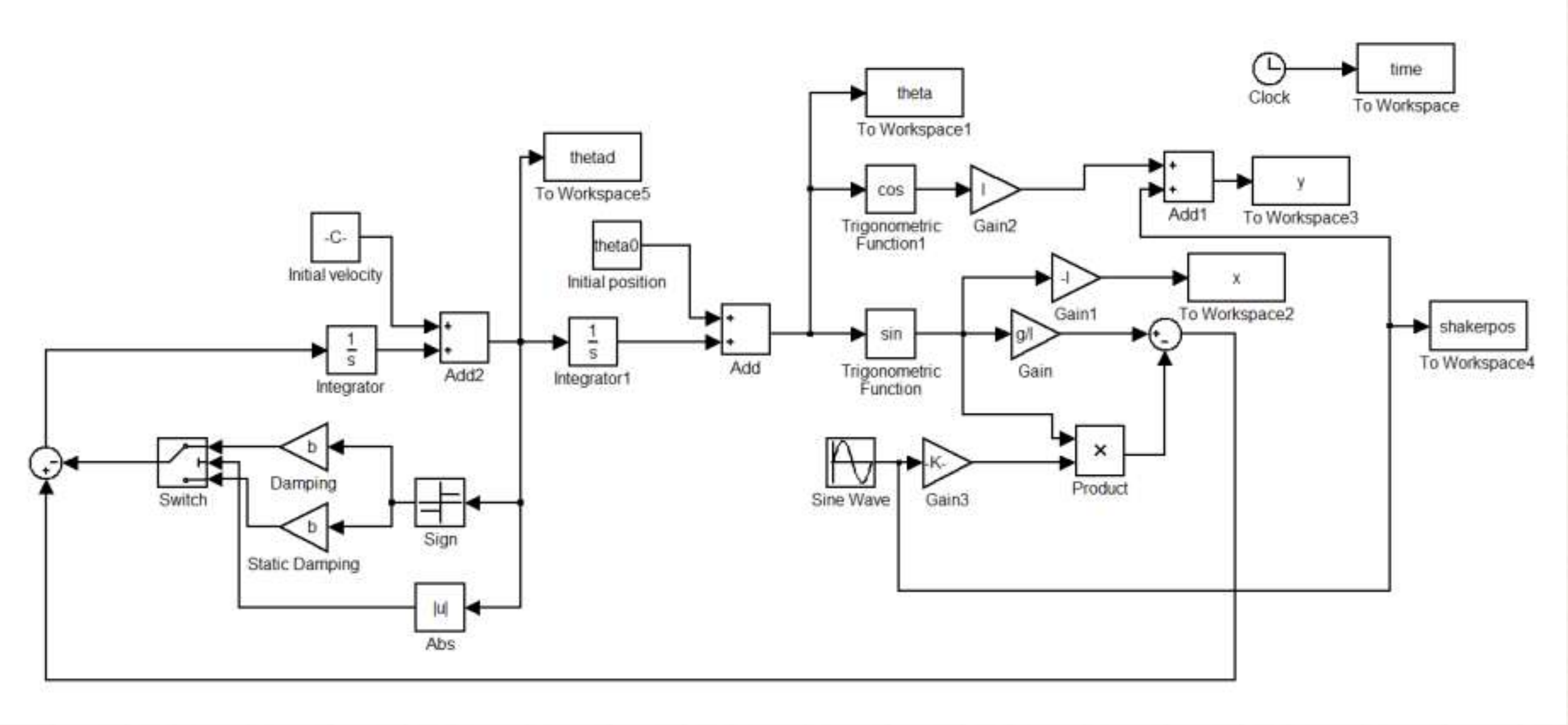
$$\eta_\pm = \dot{\theta}^*{}_\pm + \delta\theta_\pm$$

We arrive at the Mathieu equation:

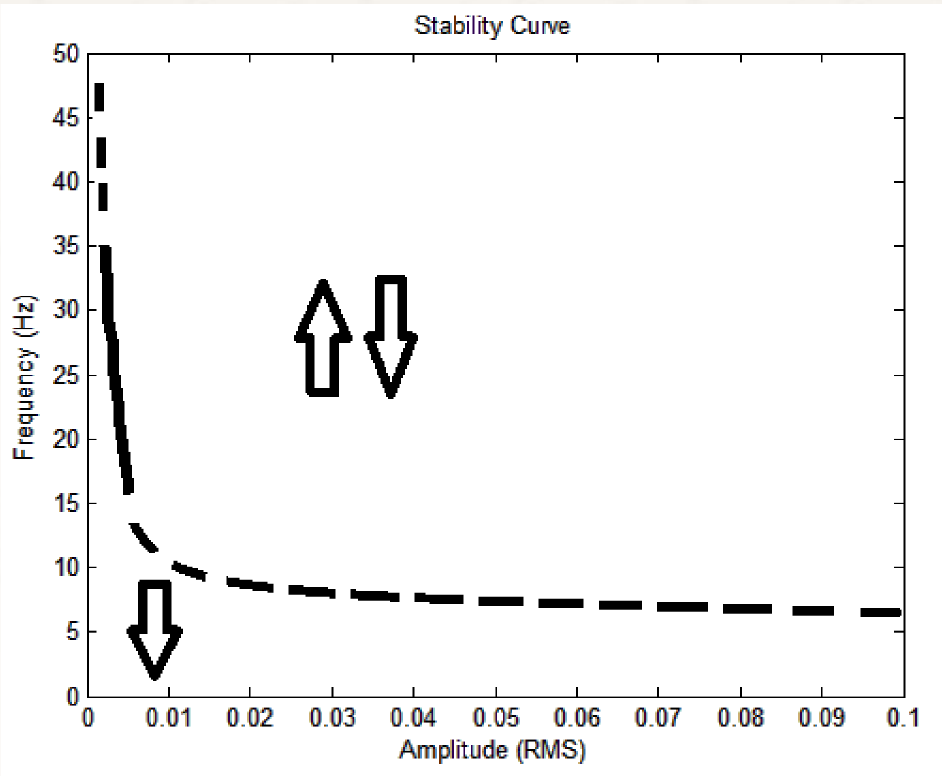
$$\delta\ddot{\theta}_\pm \pm (\beta f(\tau) - \alpha)\delta\theta_\pm = 0$$

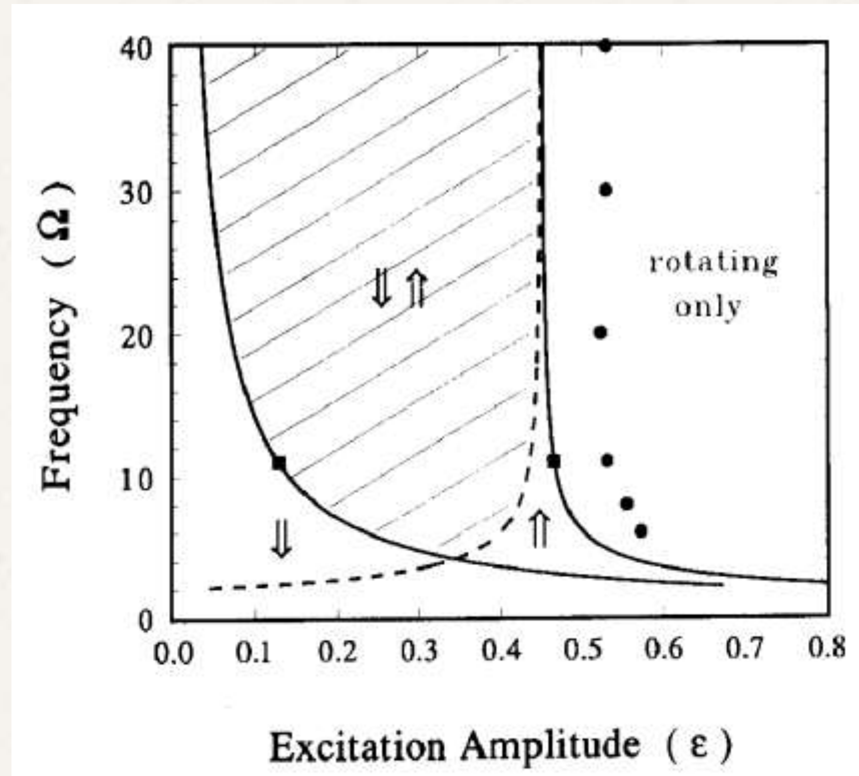
Modeling – Single Pendulum

- Using Simulink:



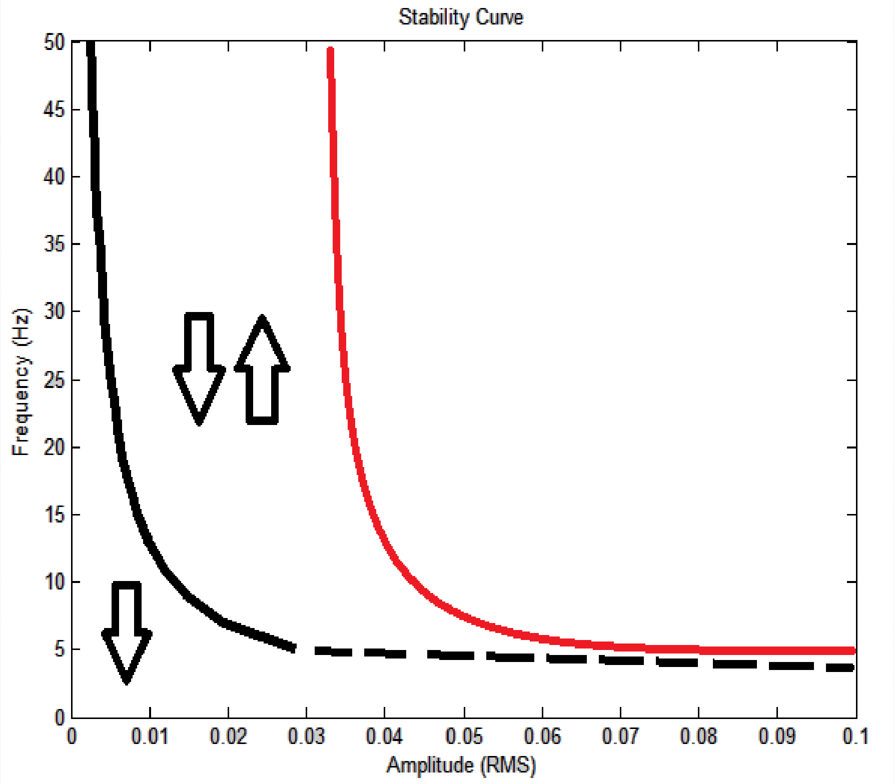
Stability Diagram - Experimental

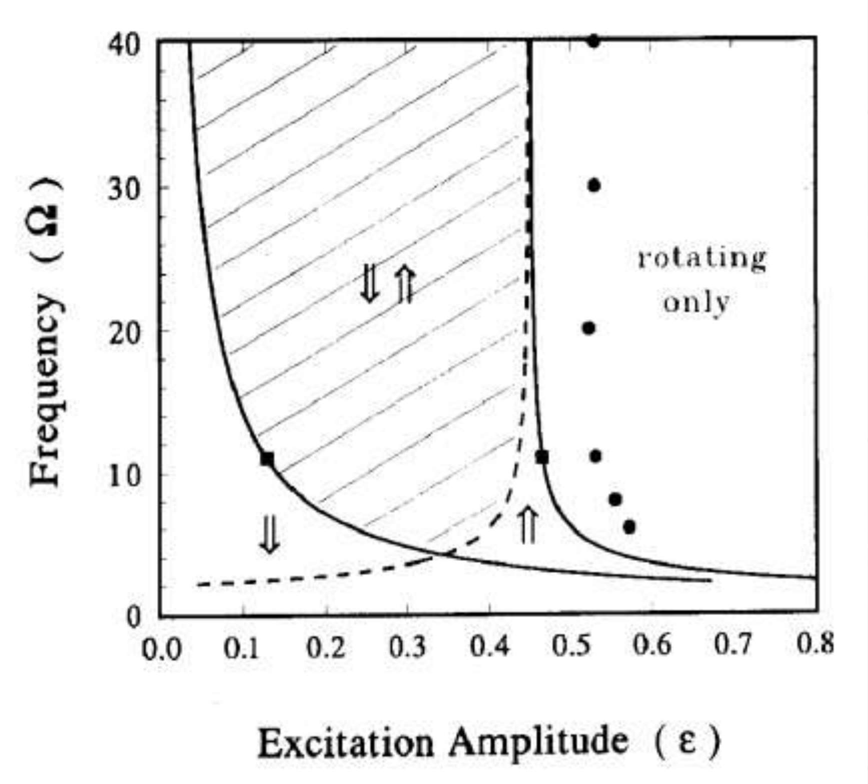




J. A. Blackburn, H. J. T. Smith, and N. Groenbech-Jensen, ‘‘Stability and Hopf bifurcations in an inverted pendulum,’’ Am. J. Phys. 60 10, 903–908 (1992).

Stability Diagram - Theoretical





J. A. Blackburn, H. J. T. Smith, and N. Groenbech-Jensen, ''Stability and Hopf bifurcations in an inverted pendulum,'' Am. J. Phys. 60 10, 903–908 (1992).

Effective Energy Potential

Effective potential with small energy losses using two-timing method.

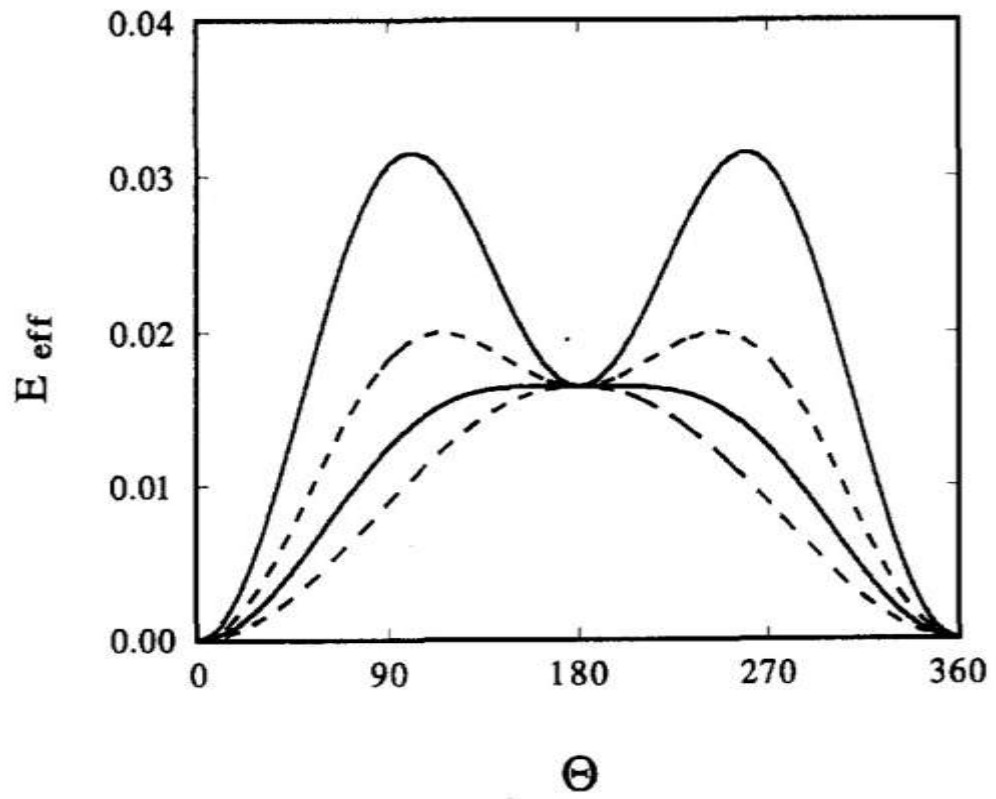
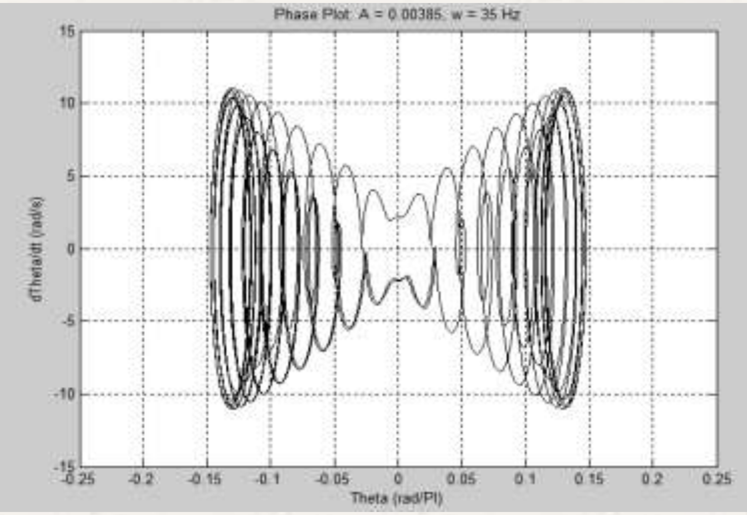
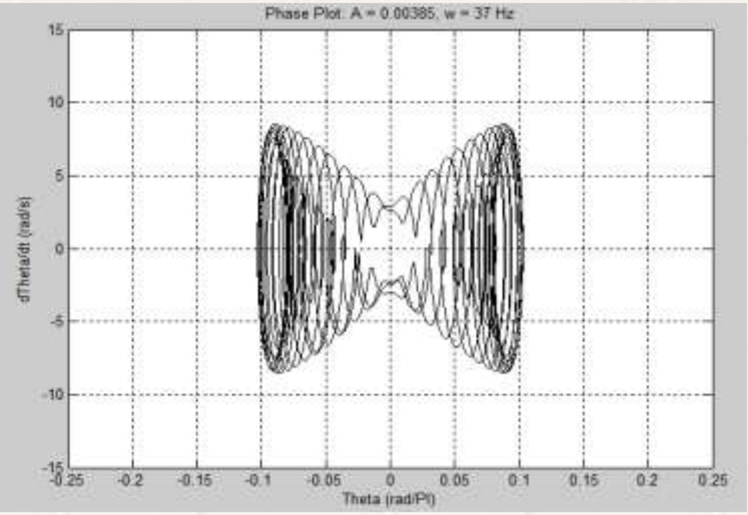


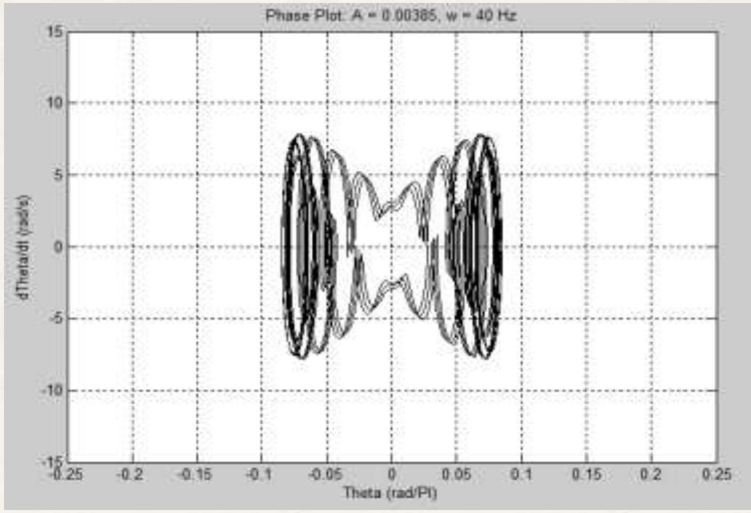
Fig. 7. Effect of damping parameter Q on the maximum release angle at which an inverted state is still reached for $\Omega = 11$.

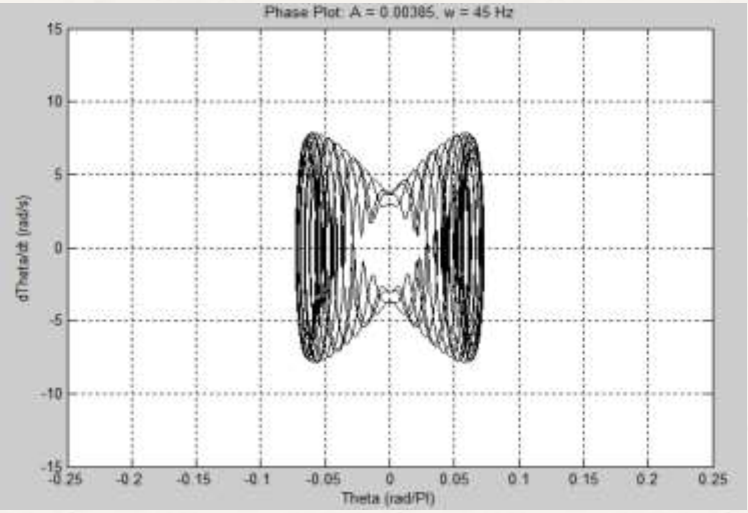
J. A. Blackburn, H. J. T. Smith, and N. Groenbech-Jensen, ‘‘Stability and Hopf bifurcations in an inverted pendulum,’’ Am. J. Phys. 60 10, 903–908 (1992).

Stability


Phase Plot: A = 0.00385, w = 35 Hz
dTheta/dt (rad/s)
Theta (rad/PI)


Phase Plot: A = 0.00385, w = 37 Hz
dTheta/dt (rad/s)
Theta (rad/PI)


Phase Plot: A = 0.00385, w = 40 Hz
dTheta/dt (rad/s)
Theta (rad/PI)


Phase Plot: A = 0.00385, w = 45 Hz
dTheta/dt (rad/s)
Theta (rad/PI)

Video: Single pendulum 25 Hz, unstable up

Video: Single pendulum at 25 Hz, stable up

Video: Single pendulum at 25 Hz, unstable up or down

Double Pendulum

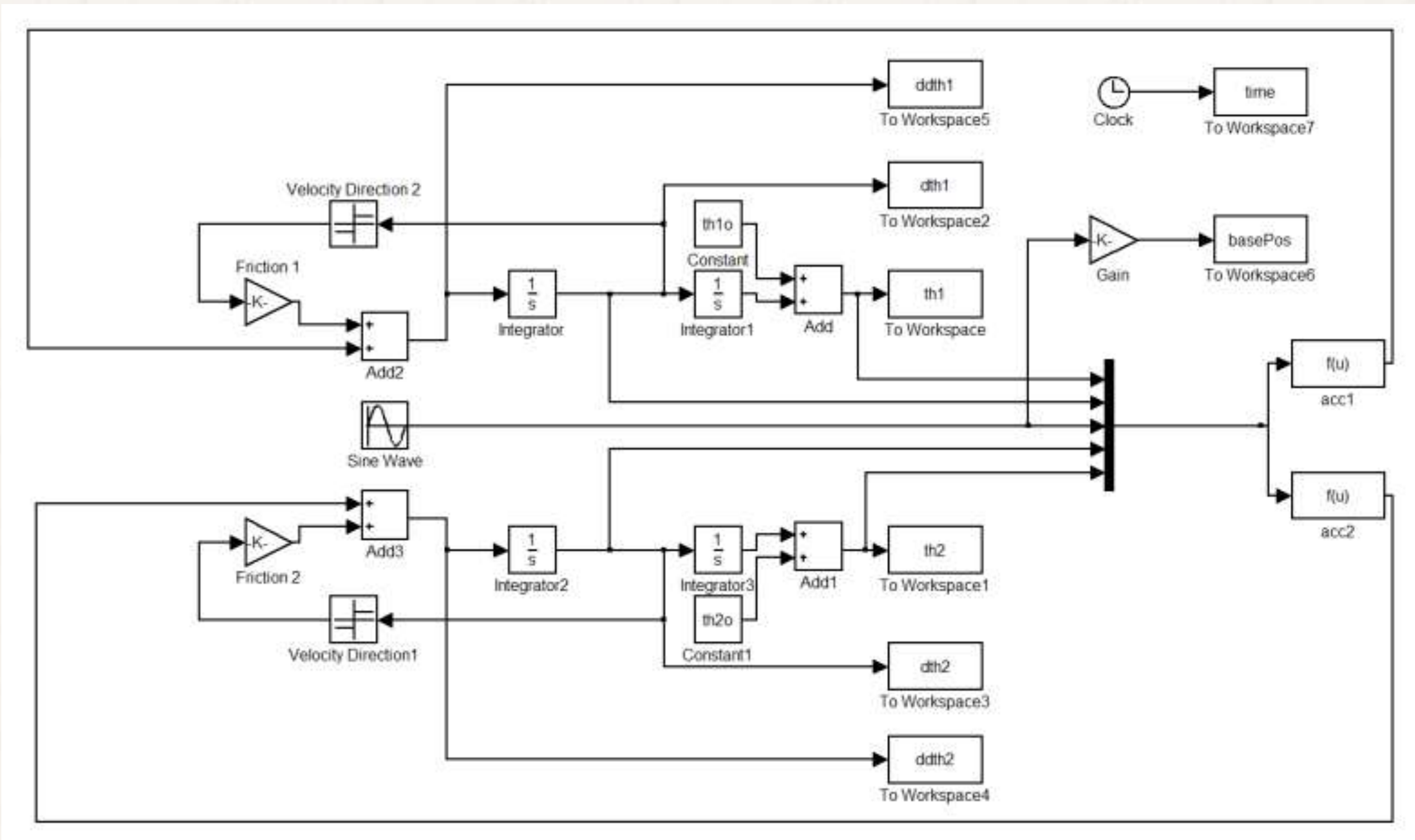
Equations of Motion:

$$\ddot{\theta}_1 = -\frac{m_2\,(L_1\,\dot{\theta}_1^{\,2}\sin(2\,\theta_1 - 2\,\theta_2) \;+ 2\,L_2\,\dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2)\,) + g\;((2\,m_1 + m_2)\sin(\theta_1) \;+ m_2\,\sin(\theta_1 - 2\,\theta_2)) - \ddot{y}\,((2\,m_1 + m_2)\sin(\theta_1) \;+ m_2\,\sin(\theta_1 - 2\,\theta_2))}{2\,L_1\,\left(m_2\sin(\theta_1 - \theta_2)^2 + m_1\right)}$$

$$\ddot{\theta}_2 = \frac{L_2\,m_2\,\dot{\theta}_2^{\,2}\sin(2\,\theta_1 - 2\,\theta_2) \;+ (m_1 + m_2)\,\left(2\,L_1\dot{\theta}_1^{\,2}\sin(\theta_1 - \theta_2) \;+ g\;(\sin(2\,\theta_1 - \theta_2) - \sin(\theta_2))\right) + \ddot{y}\,(m_1 + m_2)\,(\sin(\theta_2) - \sin(2\,\theta_1 - \theta_2))}{2\,L_2\,\left(m_2\sin(\theta_1 - \theta_2)^2 + m_1\right)}$$

Modeling – Double Pendulum

- Using Simulink:

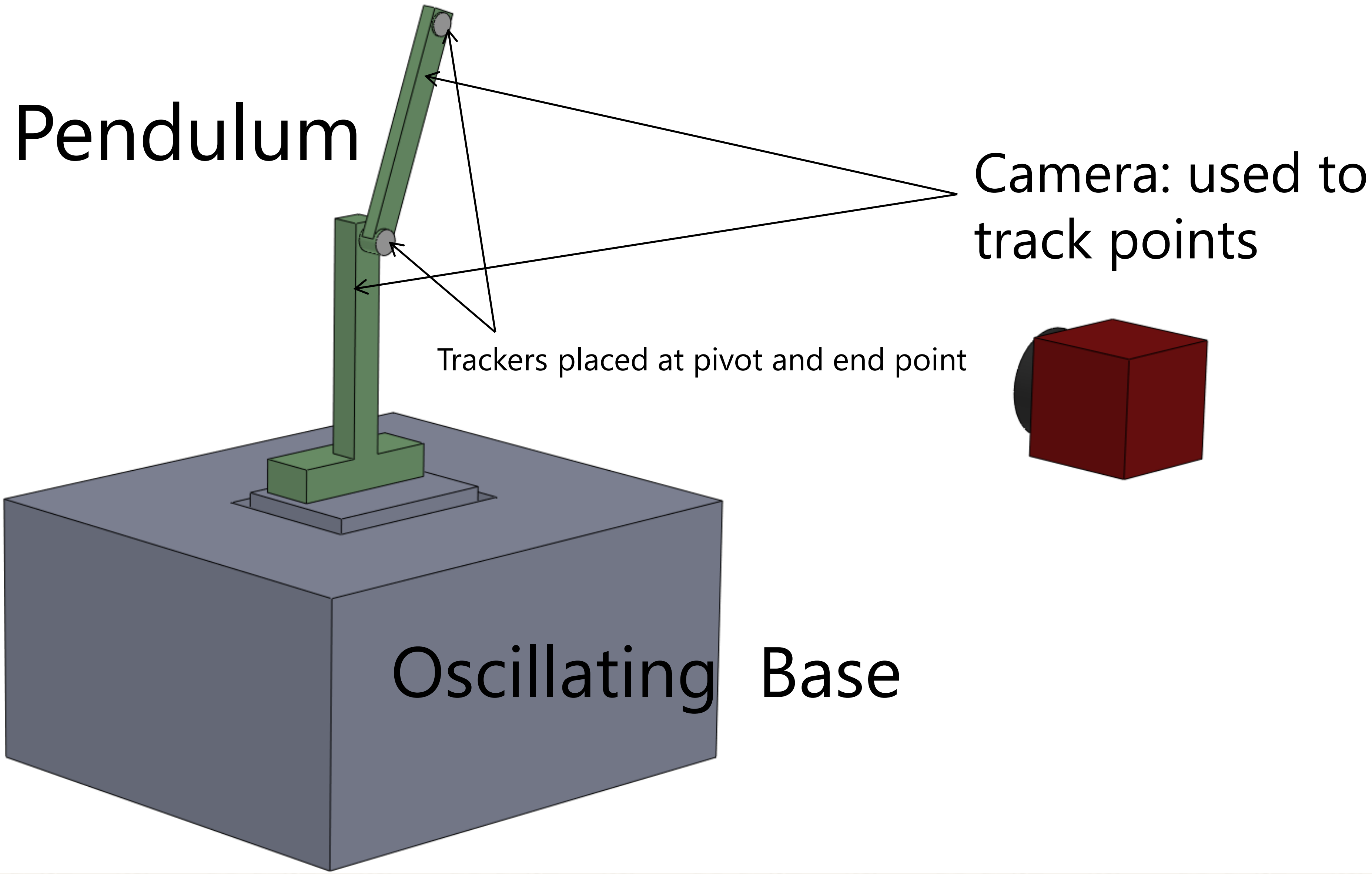


Video: Double pendulum at 50 Hz, stable

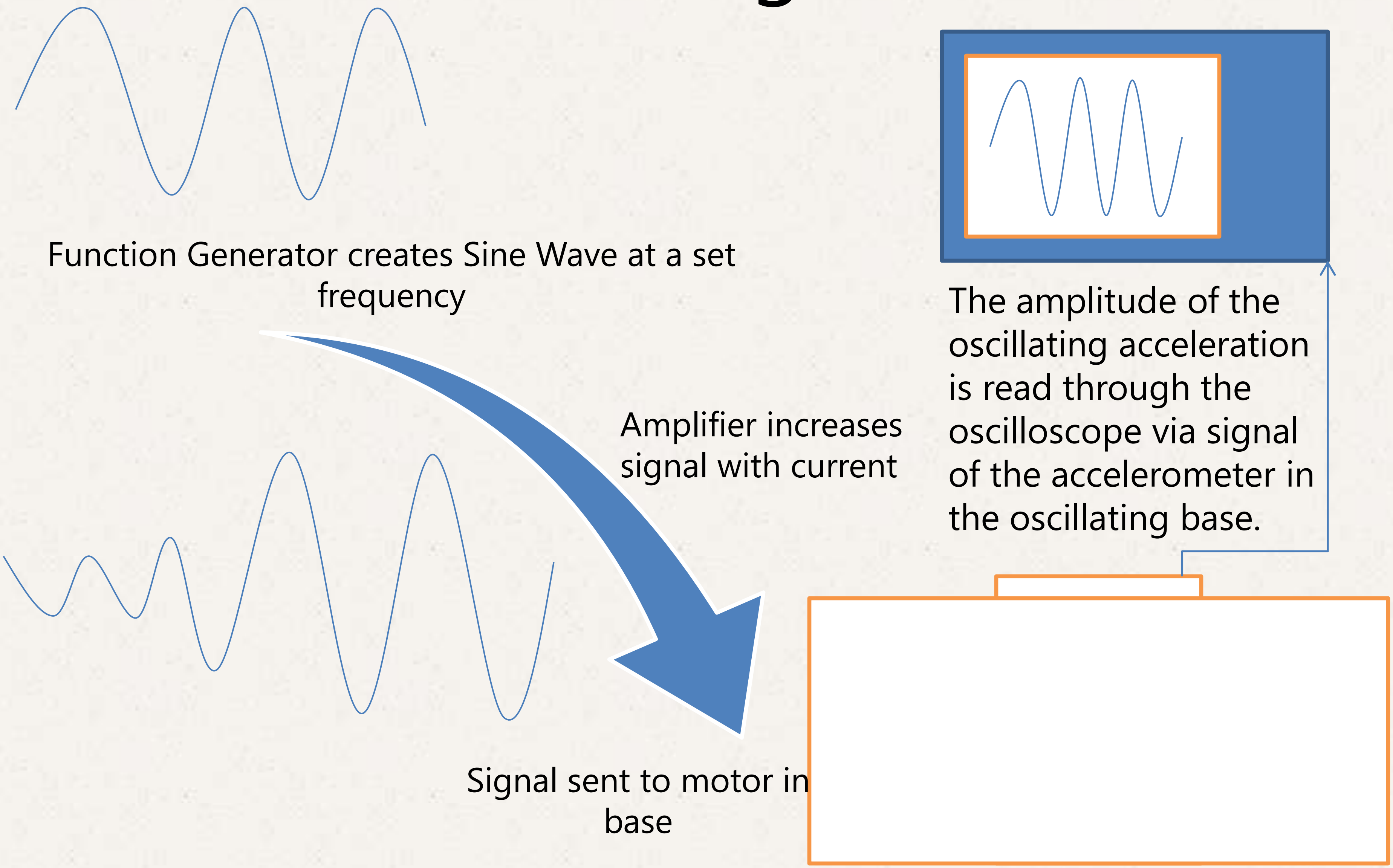
Video: Double pendulum at 50 Hz, unstable

Experimental Setup

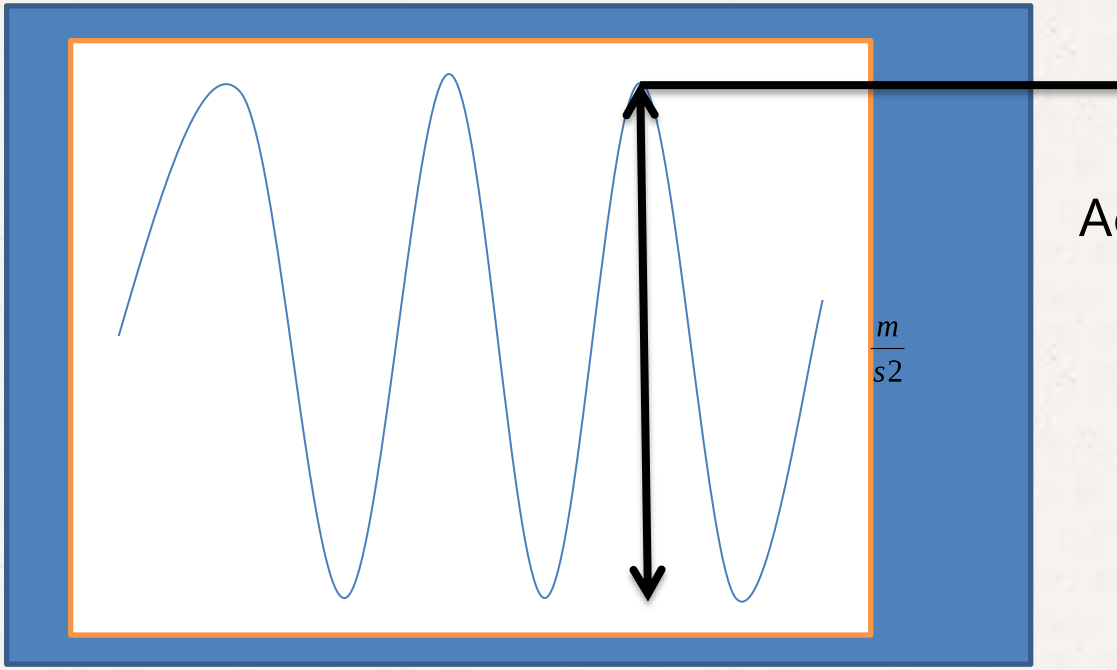
The General Setup



Oscillating Base



Forcing Amplitude from Oscilloscope



Peak to Peak Voltage: V_{pk}

Acceleration Amplitude in g:

$$A_{acc,g} = \frac{V_{pk}}{2} * 10$$

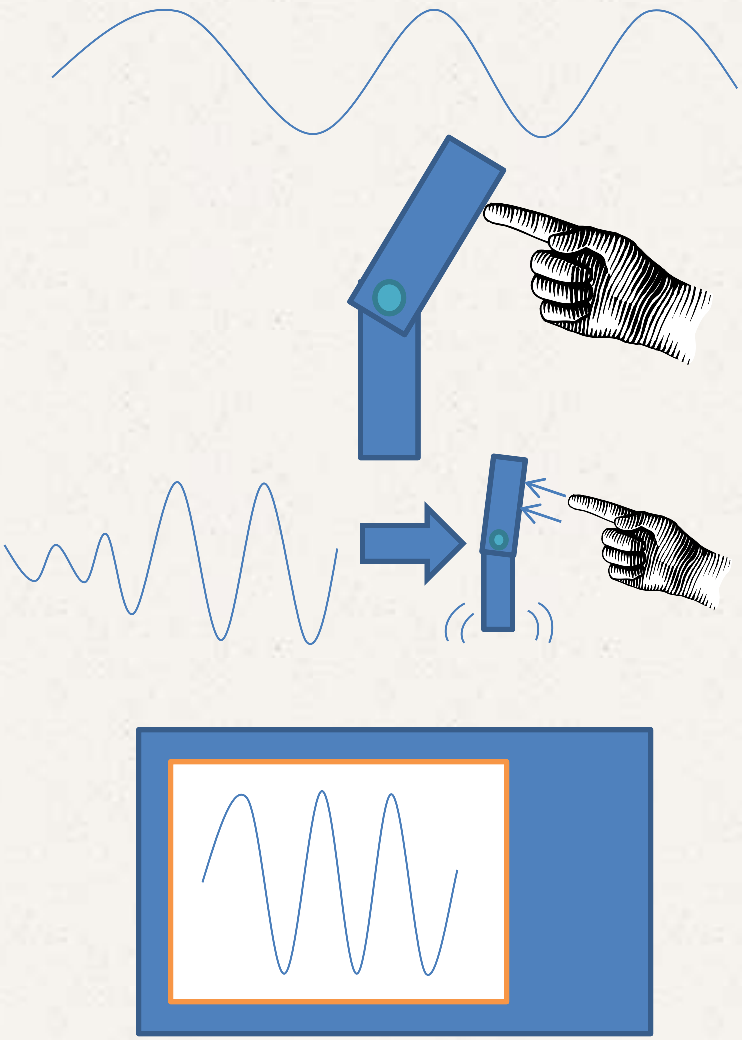
Acceleration in m/s^2:

$$A_{acc} = 9.8 * A_{acc,g}$$

Position Amplitude :

$$A_{pos} = \frac{A_{acc}}{(2\pi * Frequency)^2}$$

Stability Mapping Procedure



1. Set forcing frequency with function generator.
2. Move and support pendulum *near* top fixed point.
3. Slowly ramp up current until pendulum stabilizes at top.
4. Determine forcing amplitude from oscilloscope read out.

Tracking Setup

Trackers used:
- 10 mm white plastic balls
- White out

For 1000 fps camera, a black backdrop was required for added contrast

Camera Setup

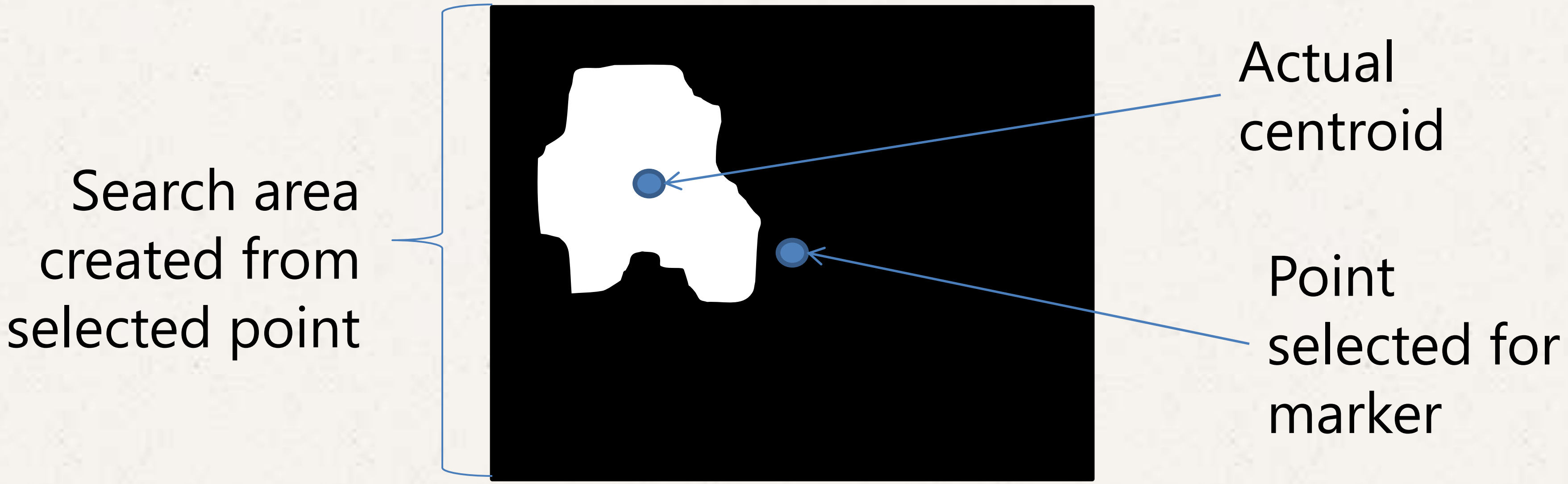
- Two different setups used:
  - PointGrey Camera:
    - ~200 fps
    - Point Tracking done in real-time in LabView
    - Only for single pendulum experiments. Tracking fails when tracking point disappears (a problem for certain pendulum setups)

Camera Setup, Continued

- Two different setups used:
  - MotionXtra
    - 1000 fps
    - Video saved to onboard drive, then sent to computer
    - Point Tracking done post-recording in Matlab.
    - Allows for tracking of fast movement (i.e. with chaotic double pendulum) as well as manual tracking at key points

Tracking in Matlab

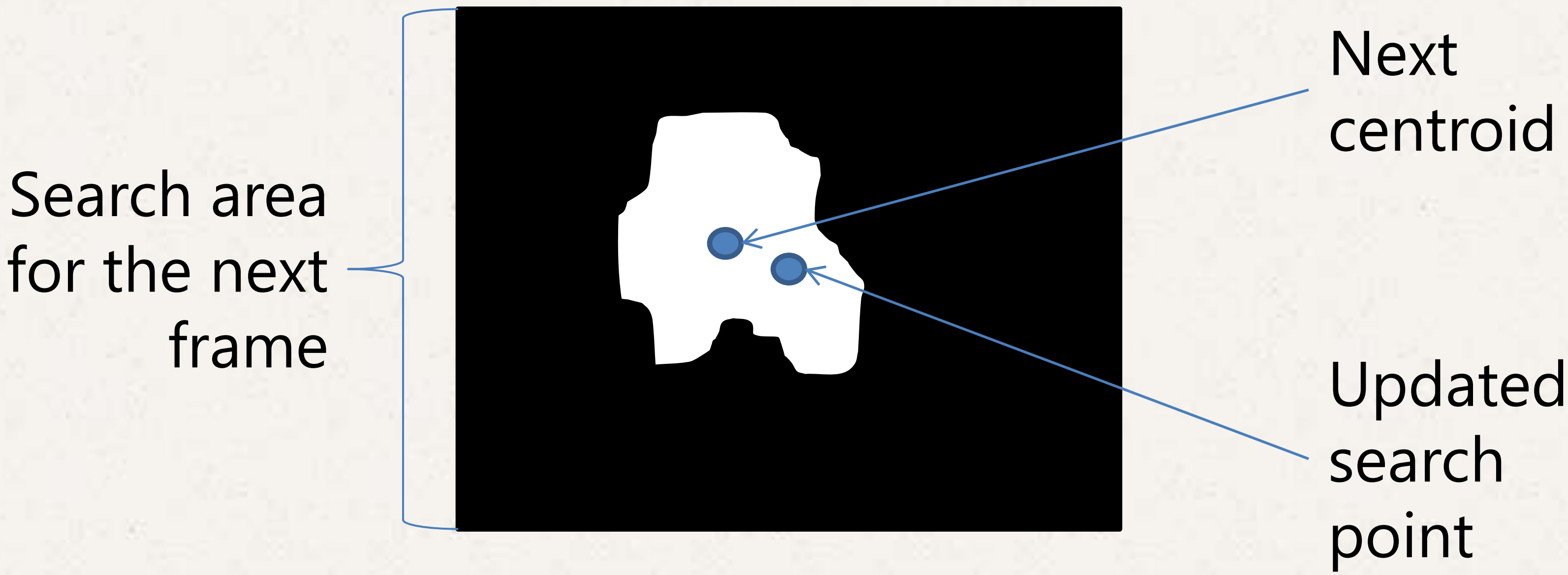
- Step one: Click on markers in the first frame.
- Step two: Threshold data.
- Step three: For a square of points around selected marker, average all of the “true” points to get centroid:



Tracking in Matlab, Continued

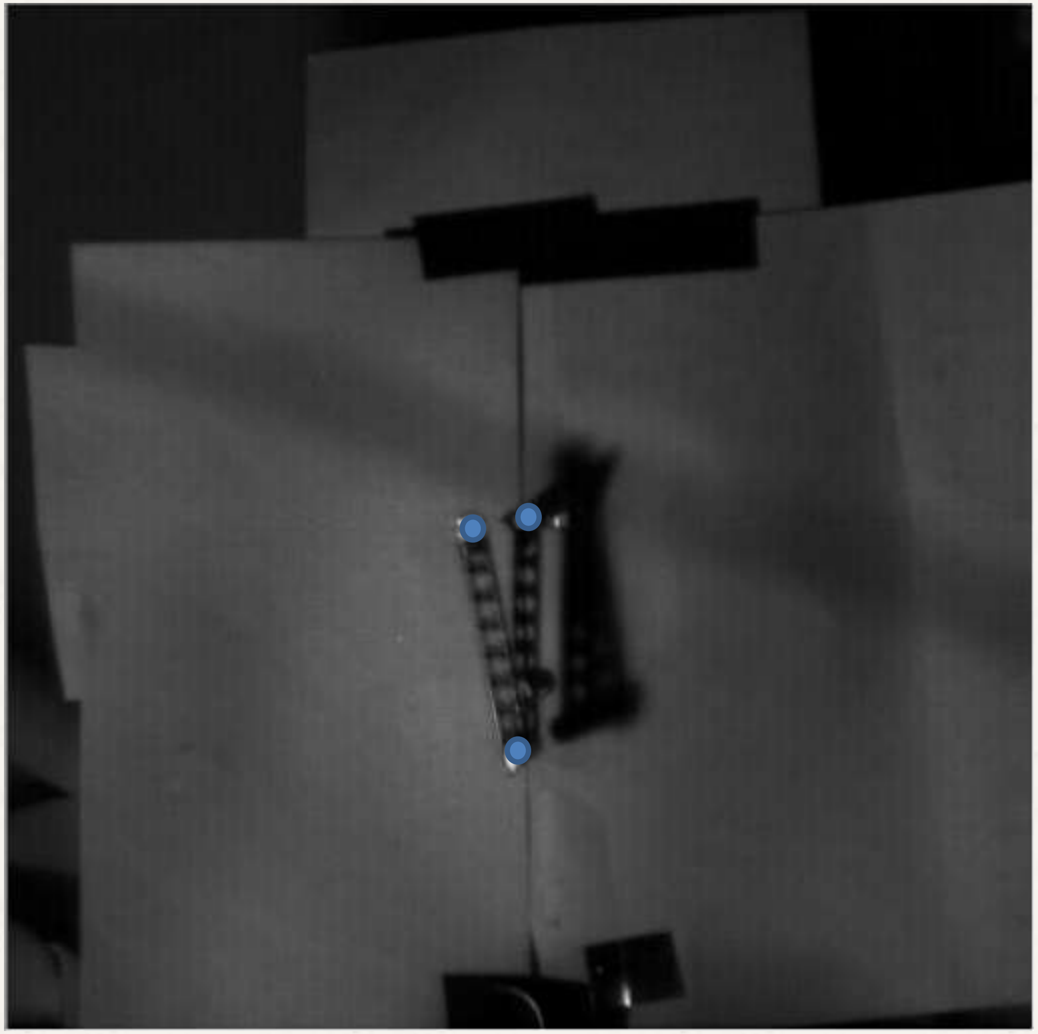
- Step four: Make the centroids the updated search points for the next frame.

REPEAT STEPS 2-4 for all frames:



Tracking the Double Pendulum

- Equal rod lengths causes **overlap** of trackers



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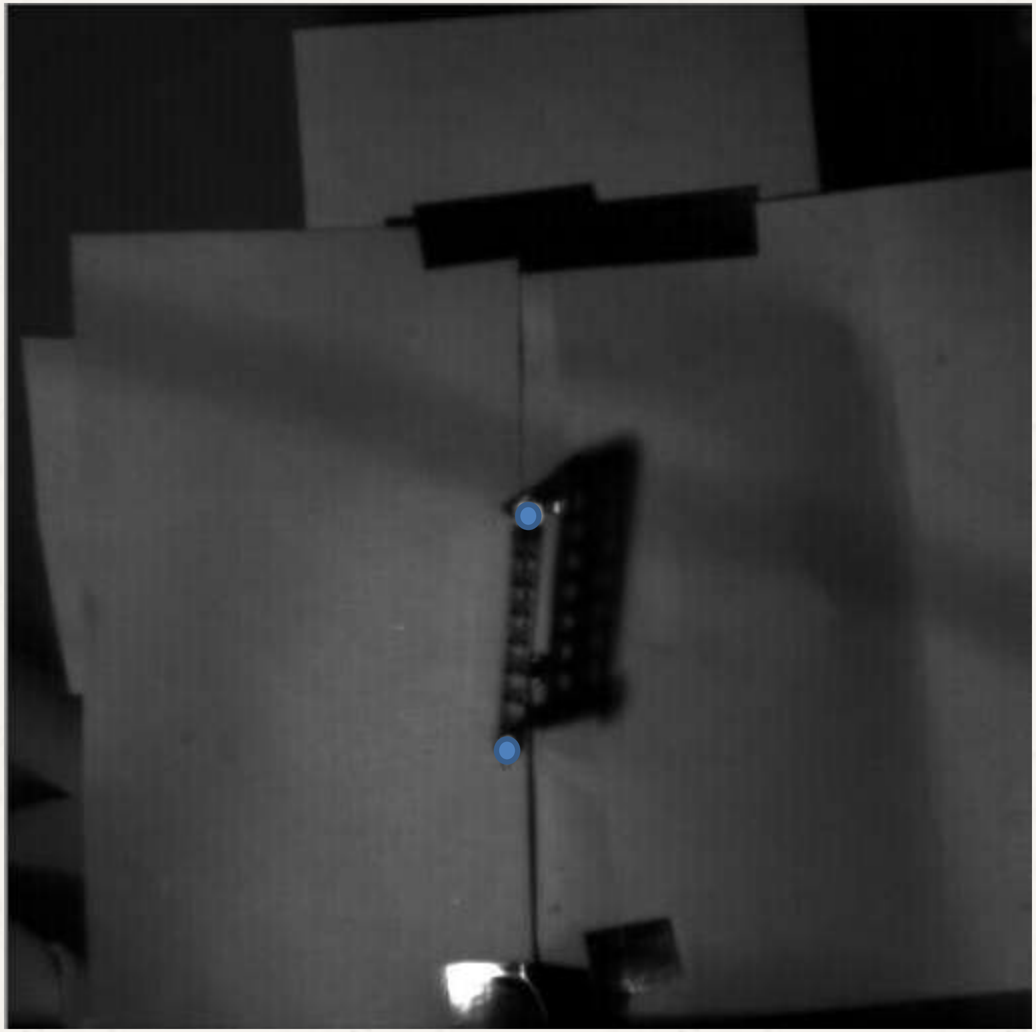

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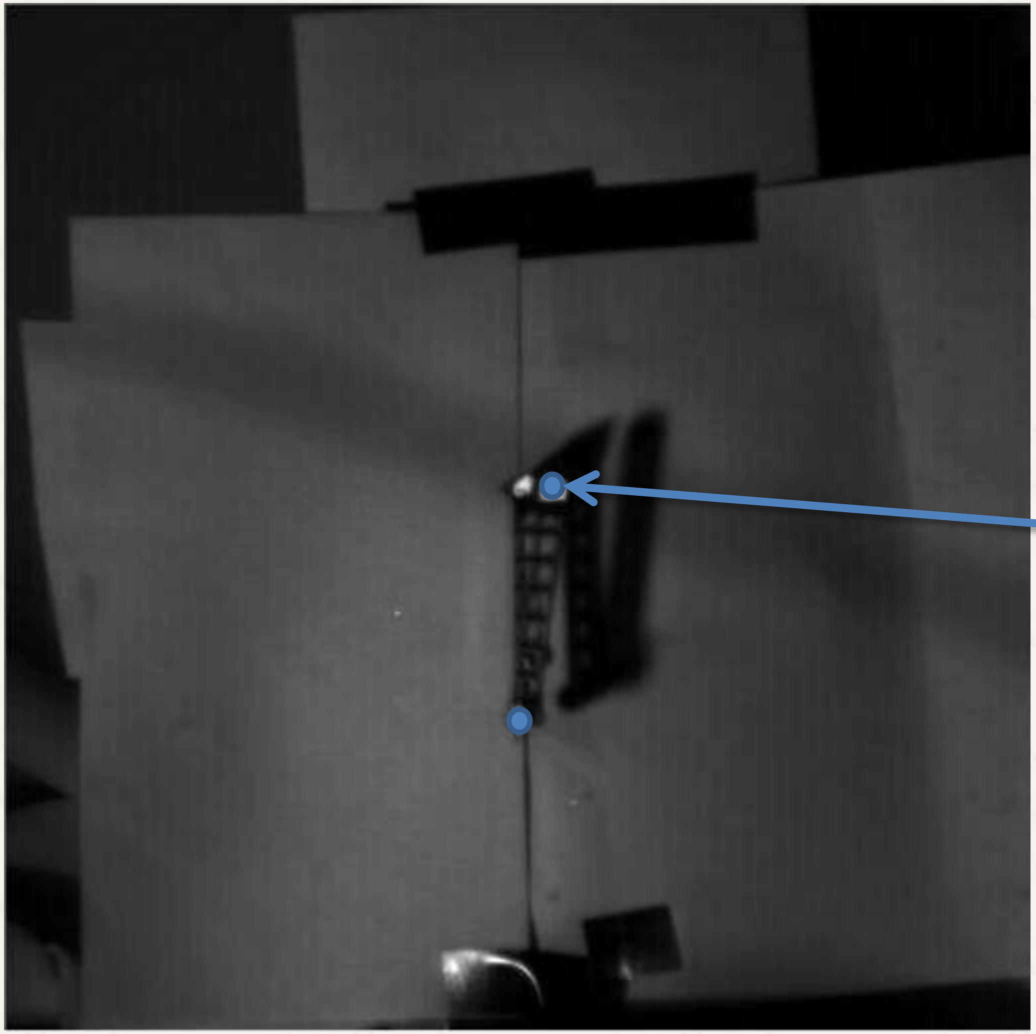
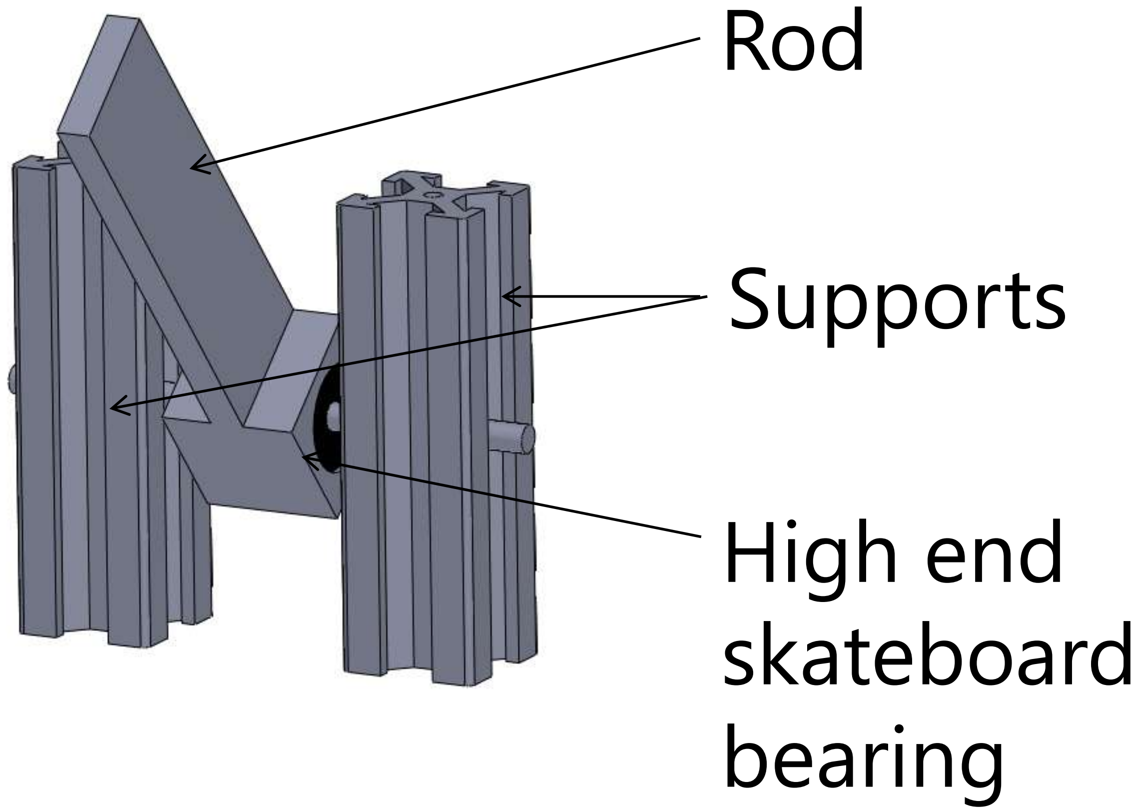


Pivot track point gets stuck on the wrong track point!

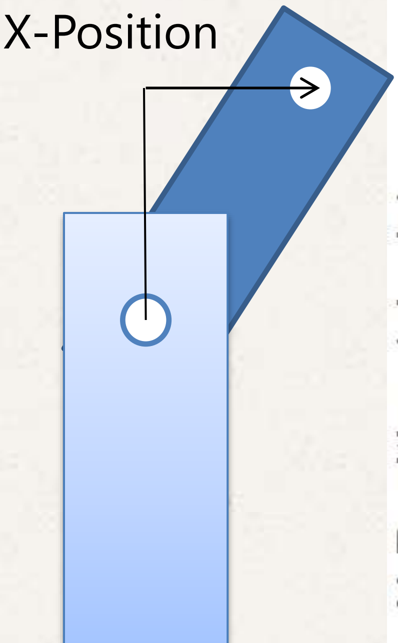
The Pendula: The Original

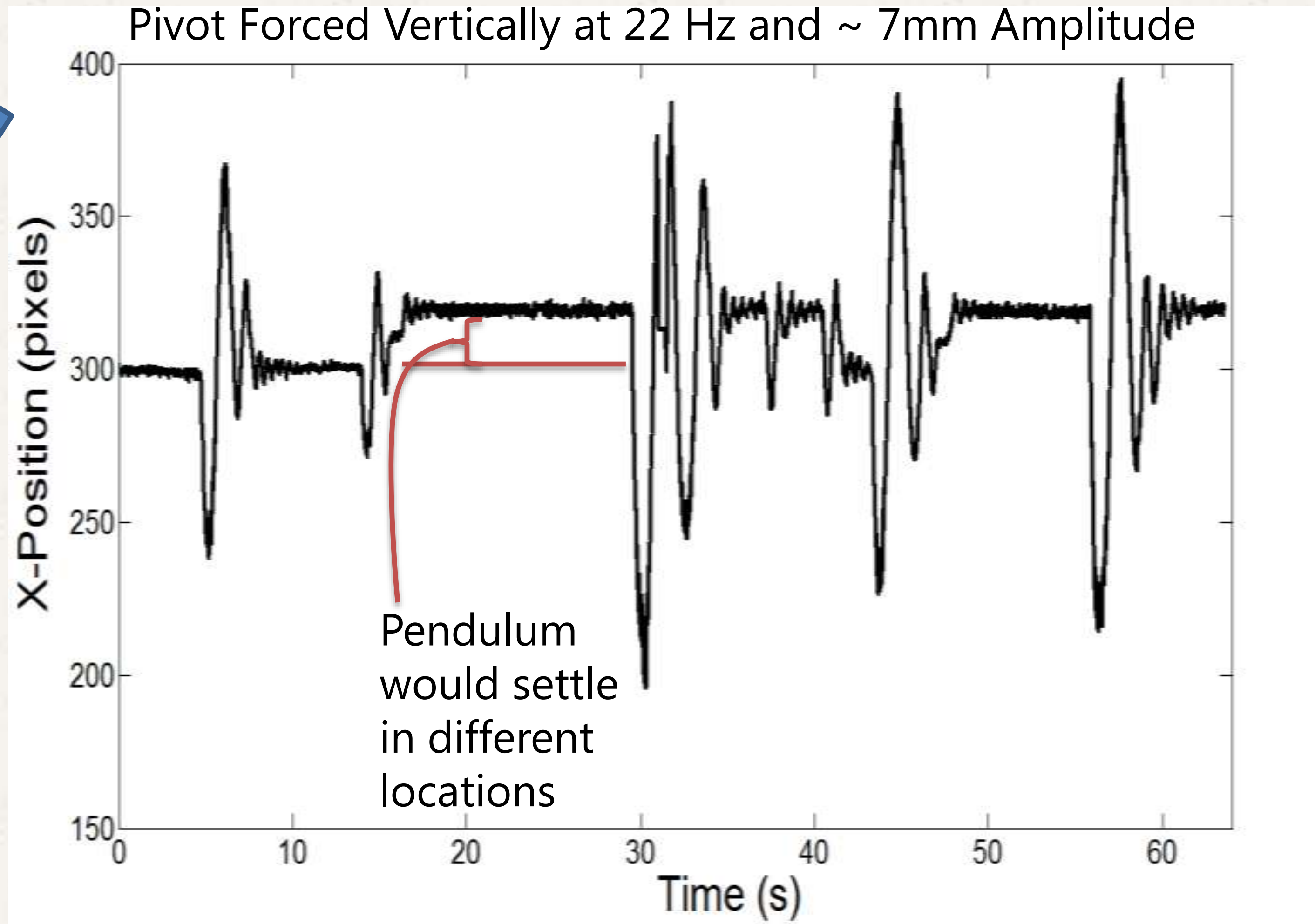
Rod Length: 10 cm



The Pendula: The Original

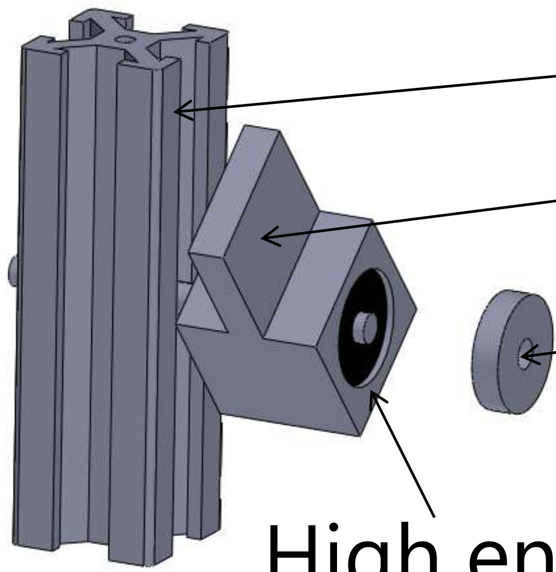
Strange high friction observed at the fixed points…





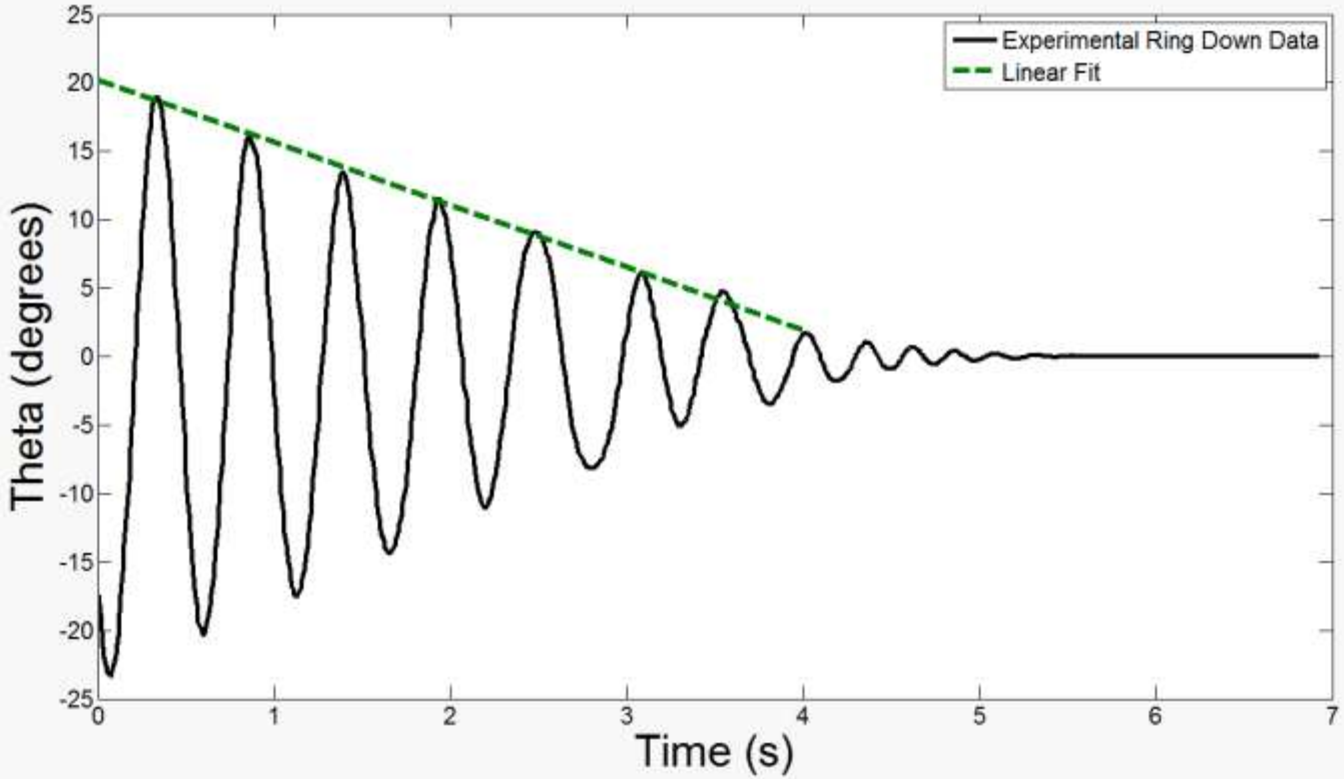
The Pendula: Modified Version

- Rod Length: 3 cm
- Noticeably decreased friction issue
- Allowed us to observe behavior at slightly higher forcing frequencies



The Pendula: Modified Version

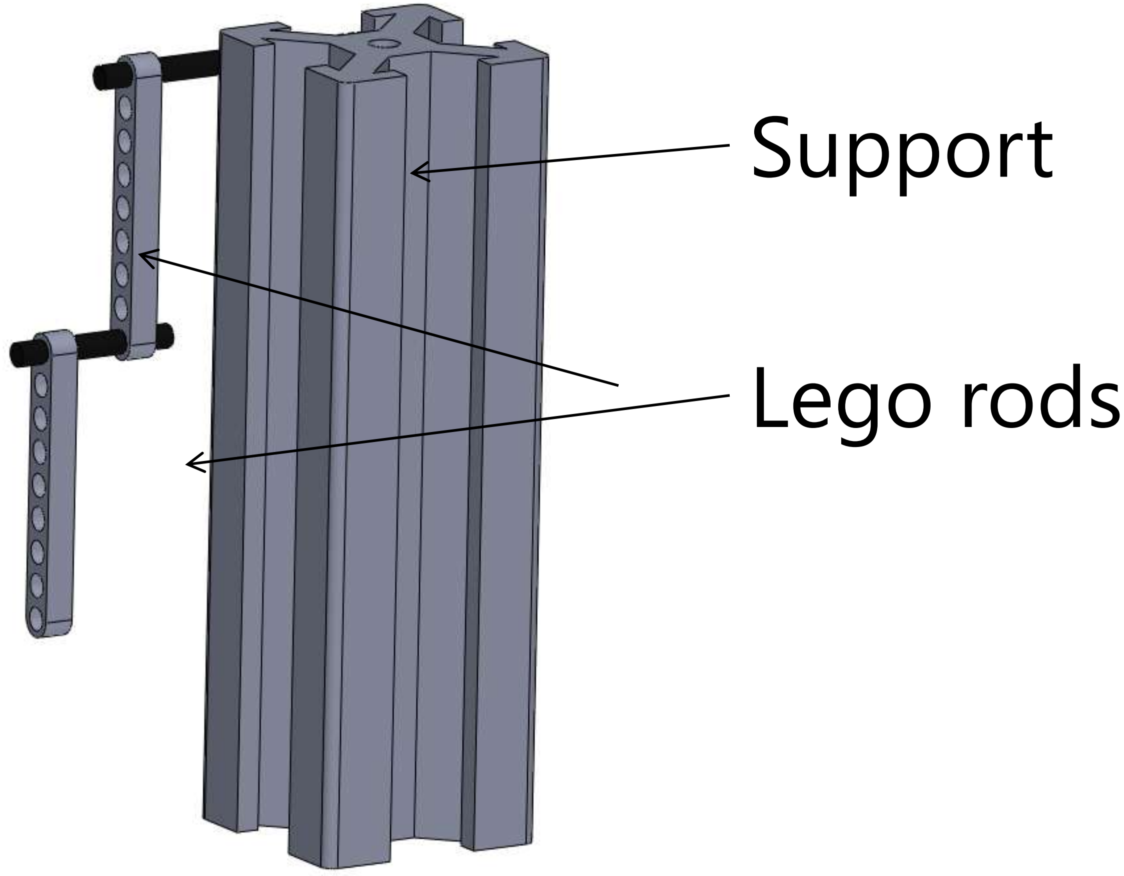
Experimental ring down at the stable fixed point (no shaking)



-Decay is **geometric** instead of **exponential**
- Suggests **frictional** damping instead of **viscous** damping

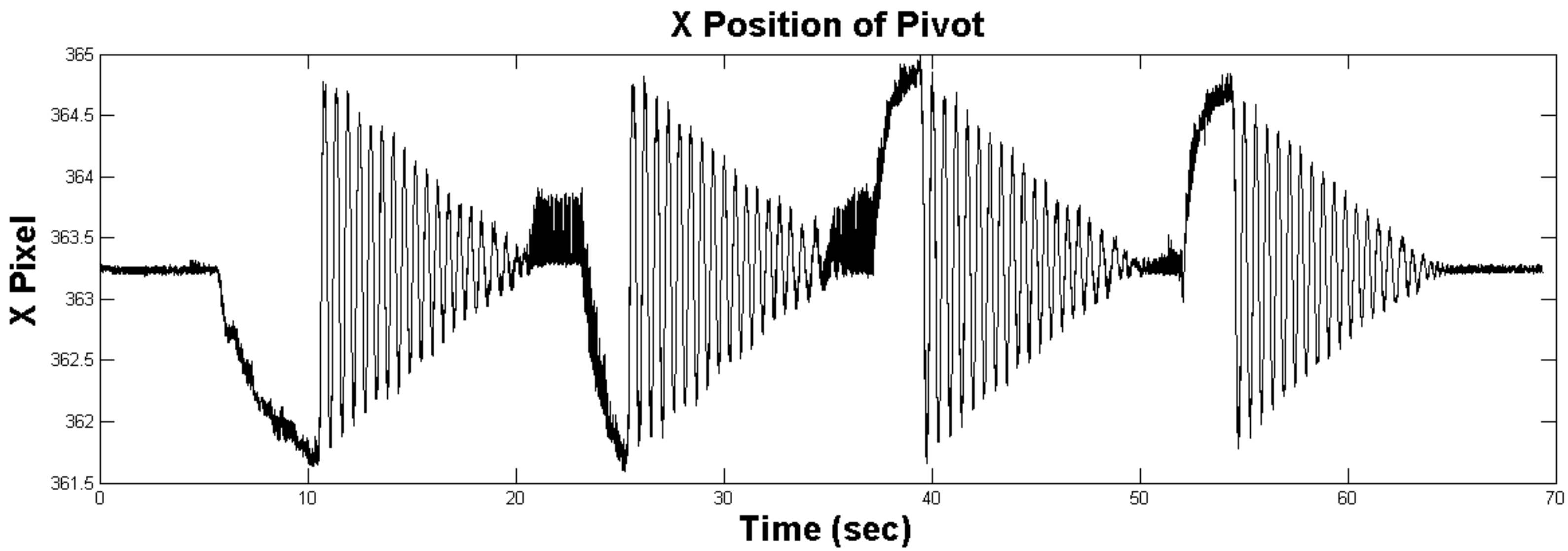
The Pendula: Lego Double

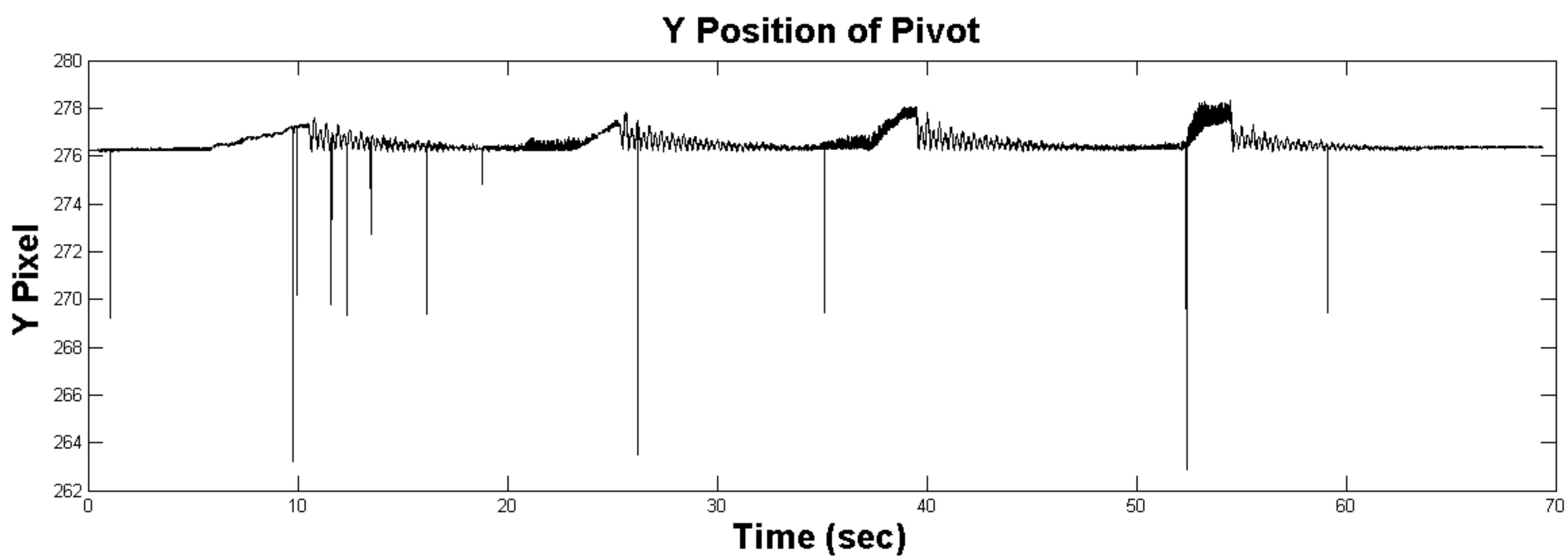
- Rod Length (6.5 cm)
- No bearings
- No strange frictional bearing issues
- Just regular, even friction
- Lesson: Legos make great pendula!
- Double Pendulum is a bit rickety



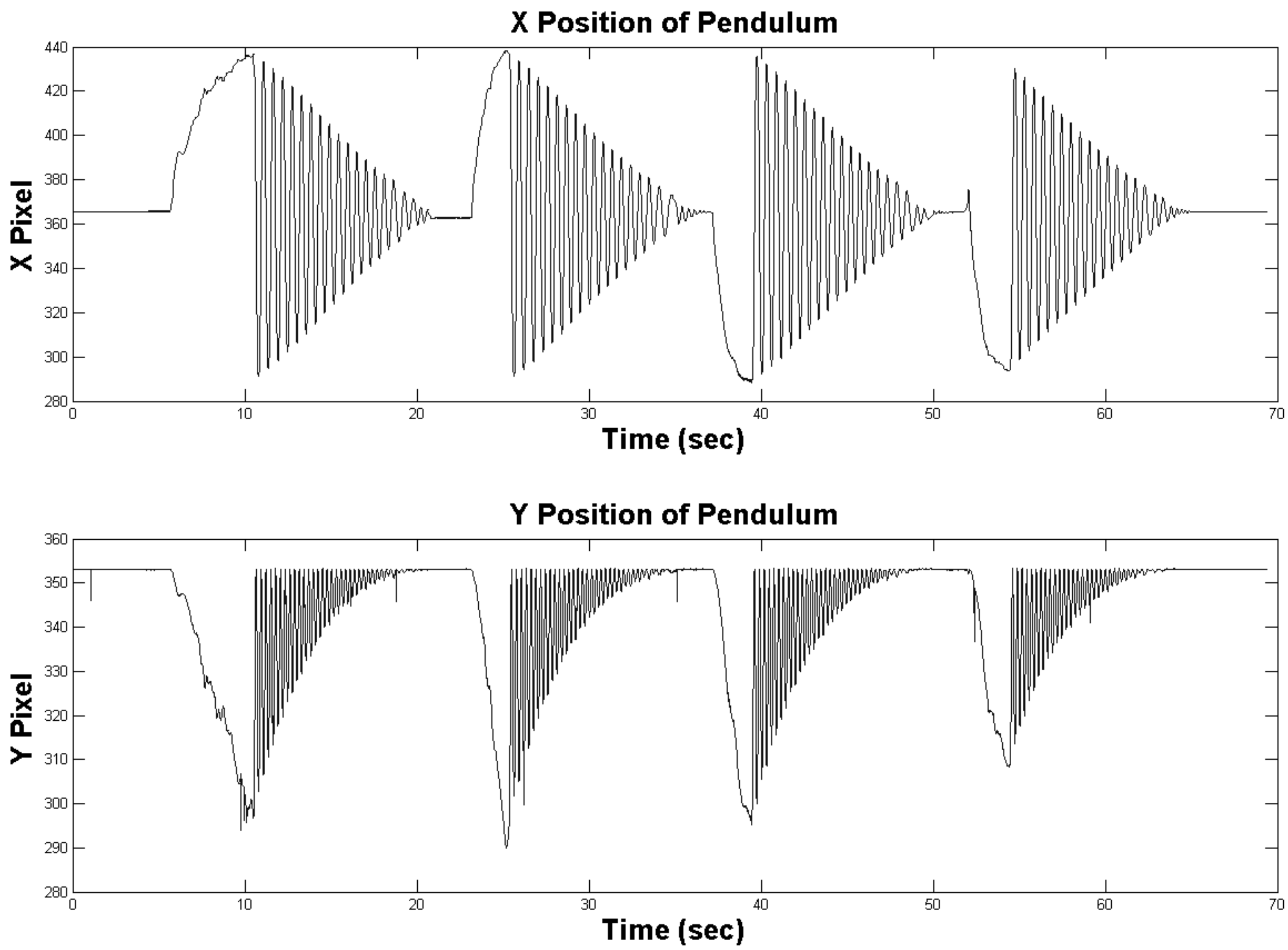
Data and Results

Tracking the Pivot

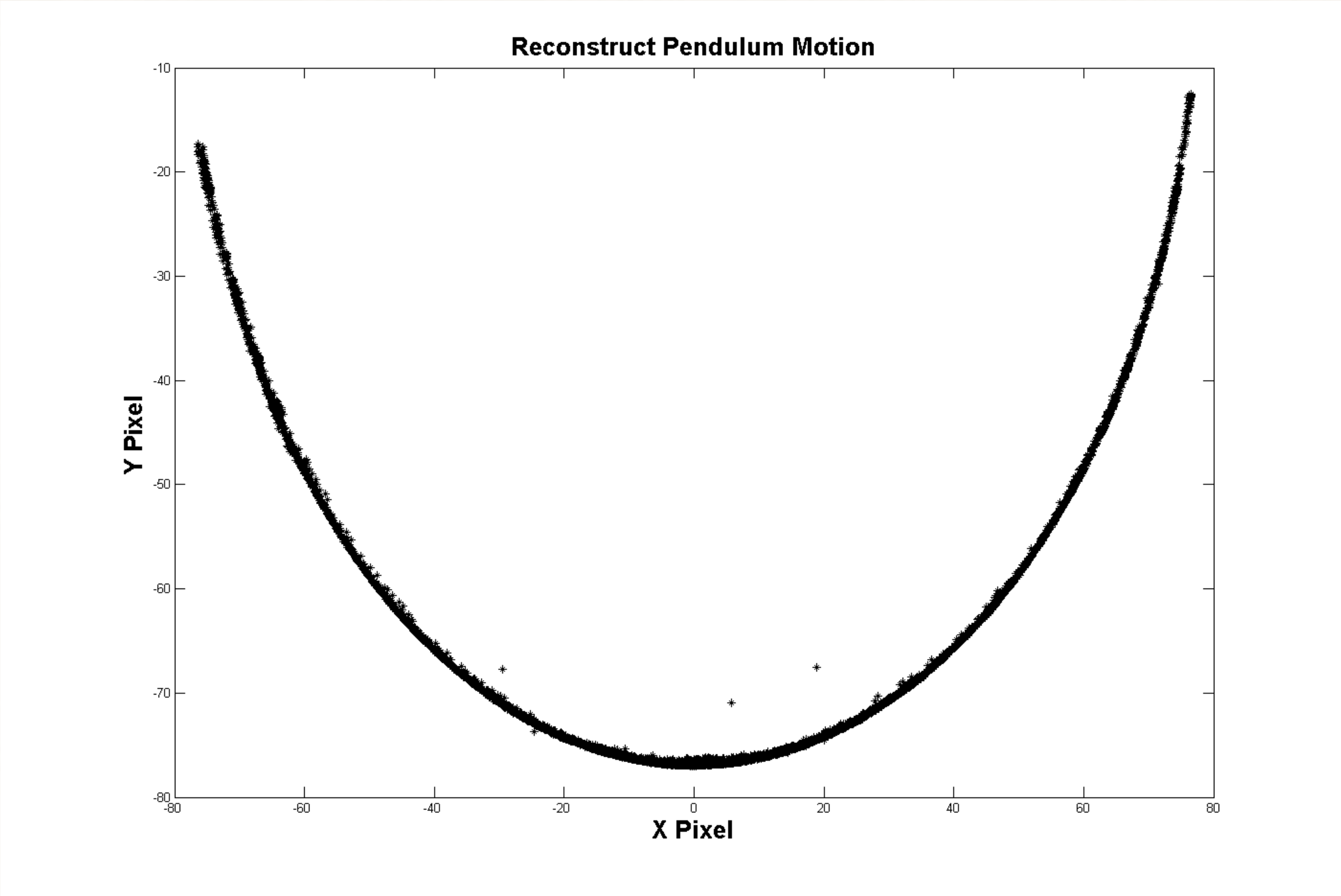




Track the Pendulum



Reconstruct Motion



Model in Physical Units

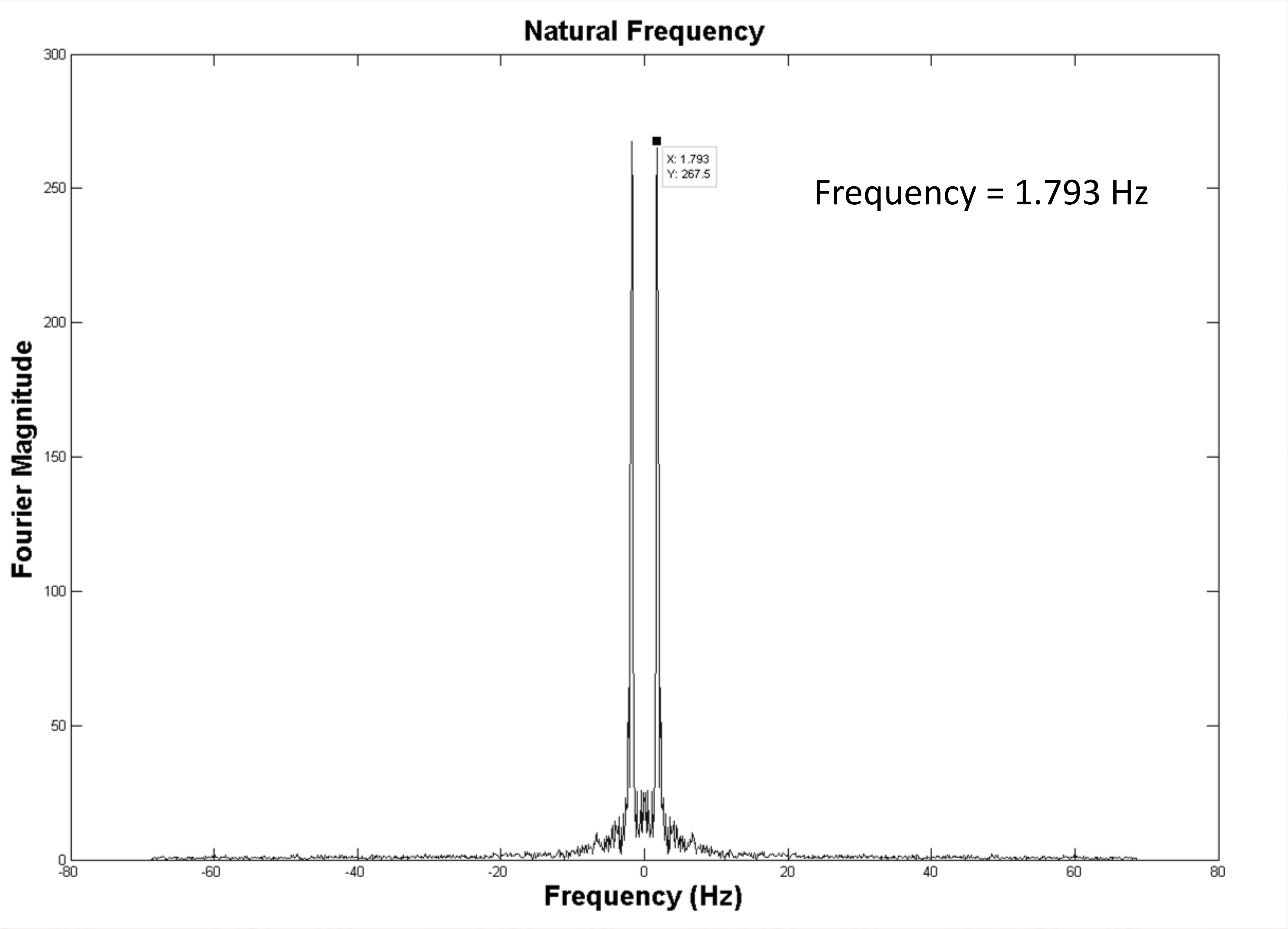
$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \Gamma\left(\hat{\dot{\theta}}\right) + \frac{\ddot{Y}}{L}\sin\theta$$ → Exact Ideal Case

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \Gamma\left(\hat{\dot{\theta}}\right)$$ → No Forcing

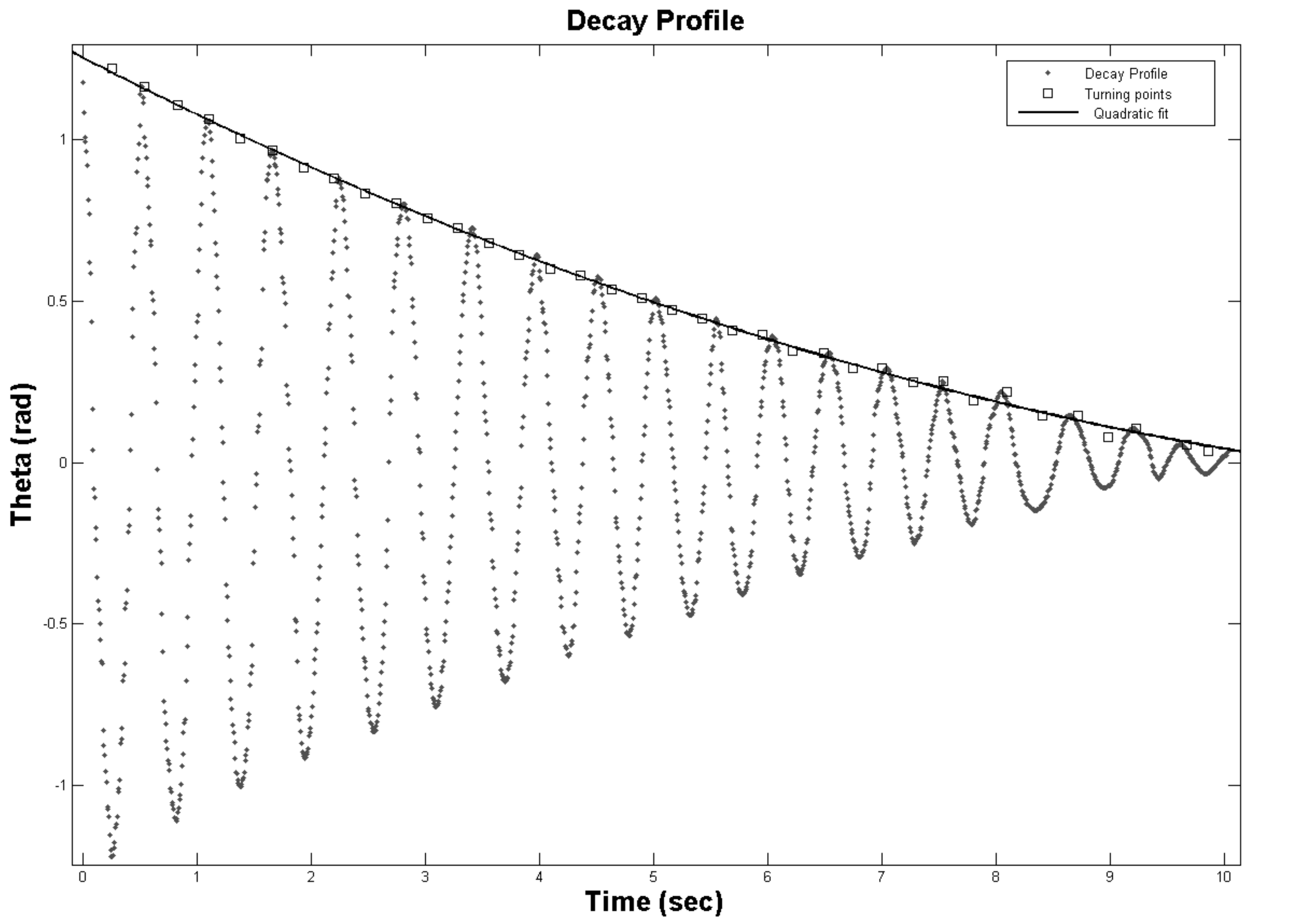
$$\ddot{\theta} = -\Omega^2\sin(\theta) - \Gamma\left(\hat{\dot{\theta}}\right)$$ → Approximation

$$\Gamma = ?$$

The Natural Frequency



Damping


Decay Profile
Decay Profile
Turning points
Quadratic fit
Theta (rad)
Time (sec)
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
10

Finding Γ

- How can we relate Γ to the decreasing Amplitude??
- Work done by friction equals the loss in potential energy i.e. Amplitude
- Small Angle approximation yields the following expression

$$\Gamma = \pi^2 \frac{(\Delta\theta_{max})}{T}\left(\frac{1}{T}\right) = \pi^2 mf$$

- Experiment: Large Amplitude, Small Amplitudes specially distorted

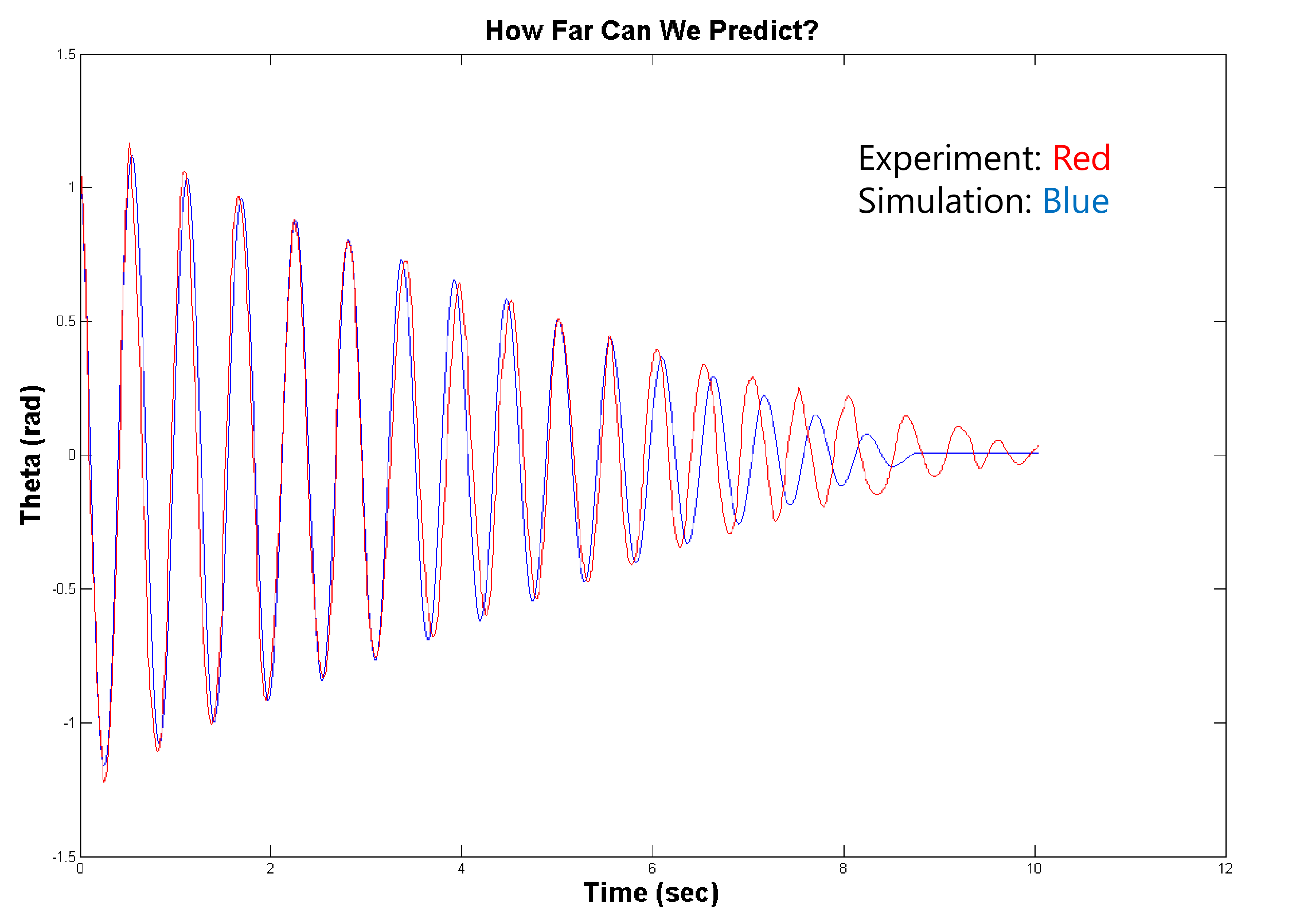
Determine Parameters

- Determined the frequency from decay data
- Fit the first oscillation to a sin/cos and get the initial theta and omega

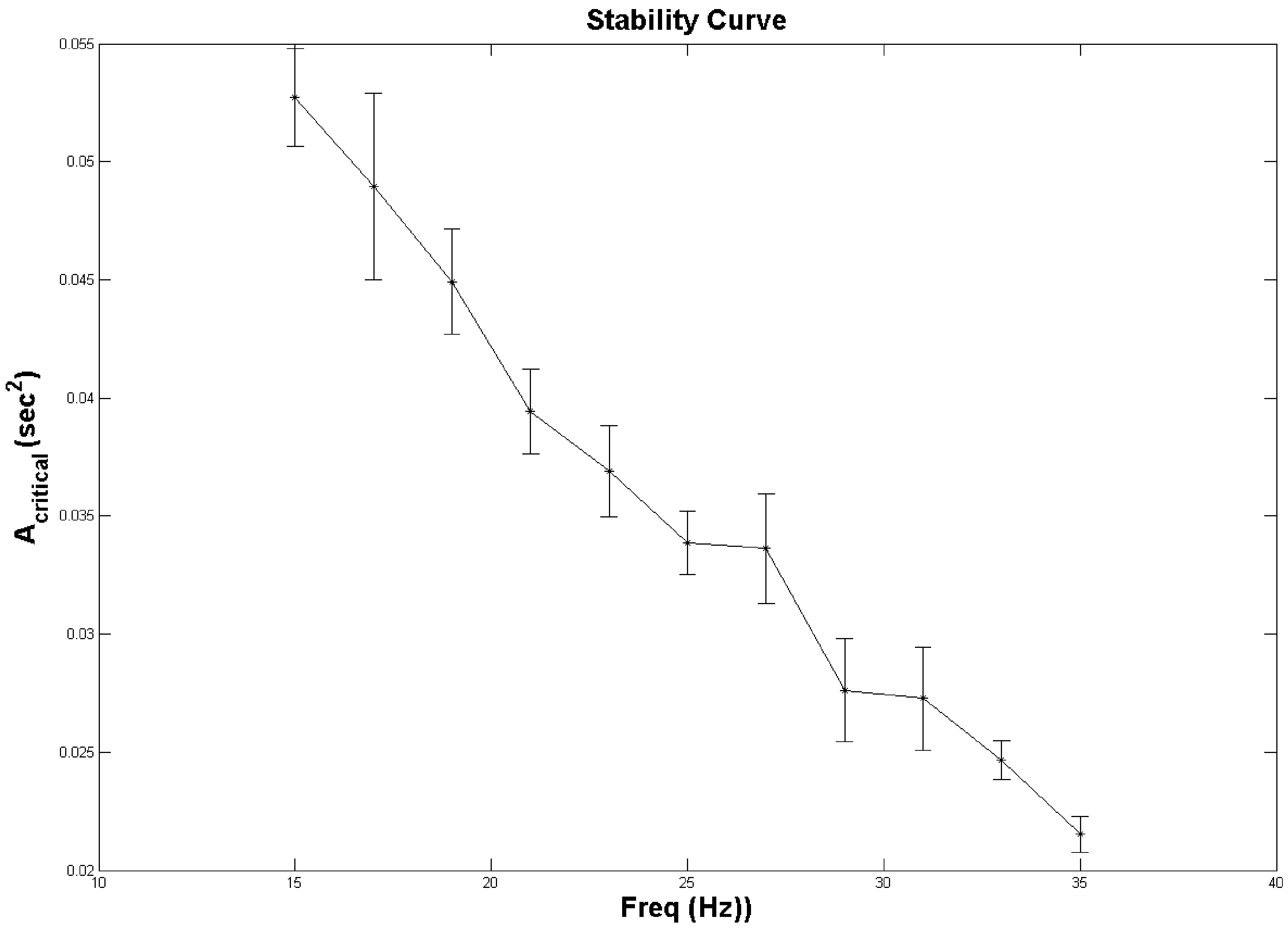
- A Linear/Quadratic fit of the decay turning amplitudes gives $\Gamma \sim 3$
- Initial Conditions: $\theta_0 = 1.1 \quad \omega_0 = -5.6$
- Tweak frequency and Friction

$$f = 1.87 Hz \quad \Gamma \sim 2.5$$

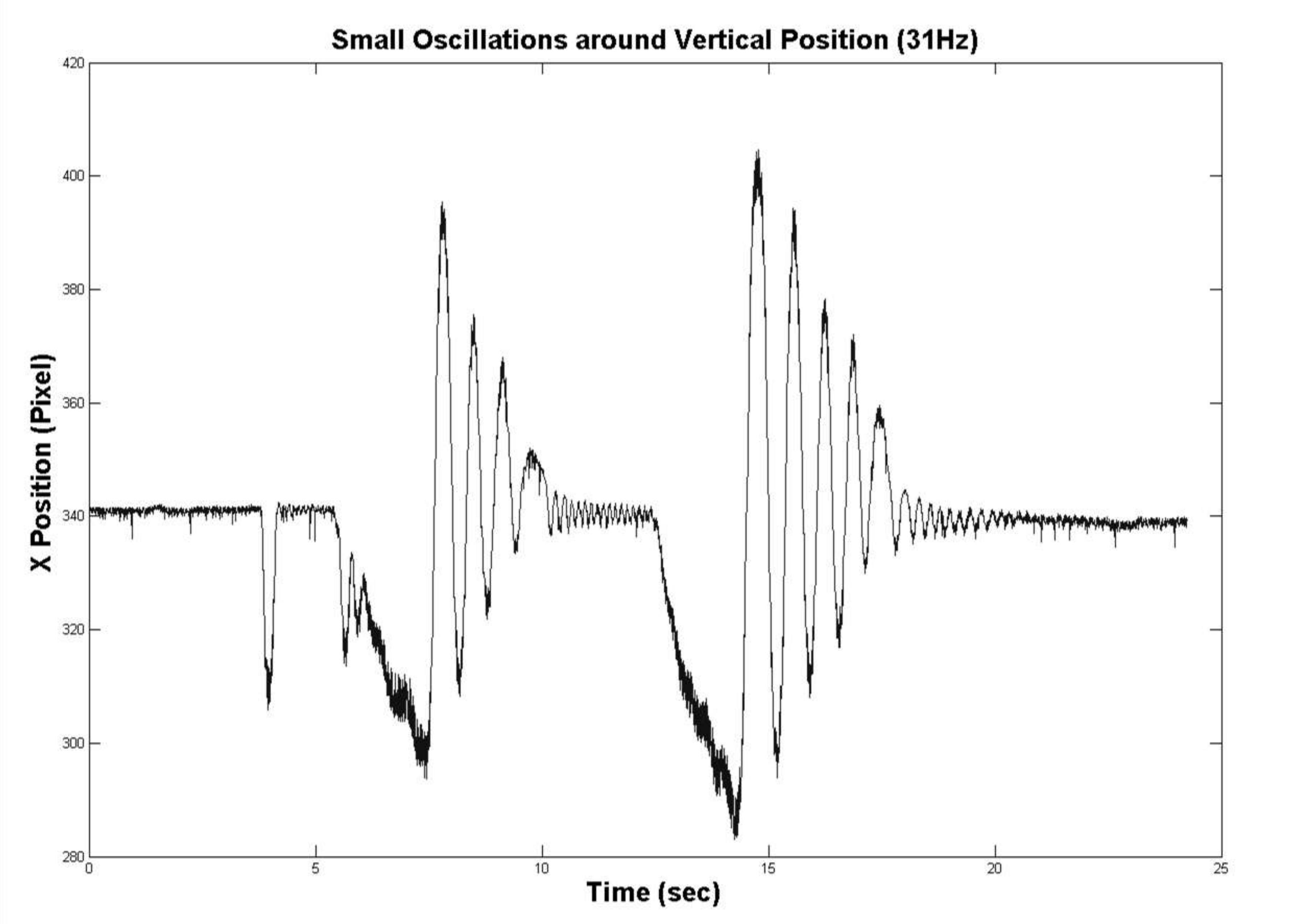
Simulation Vs Experiment



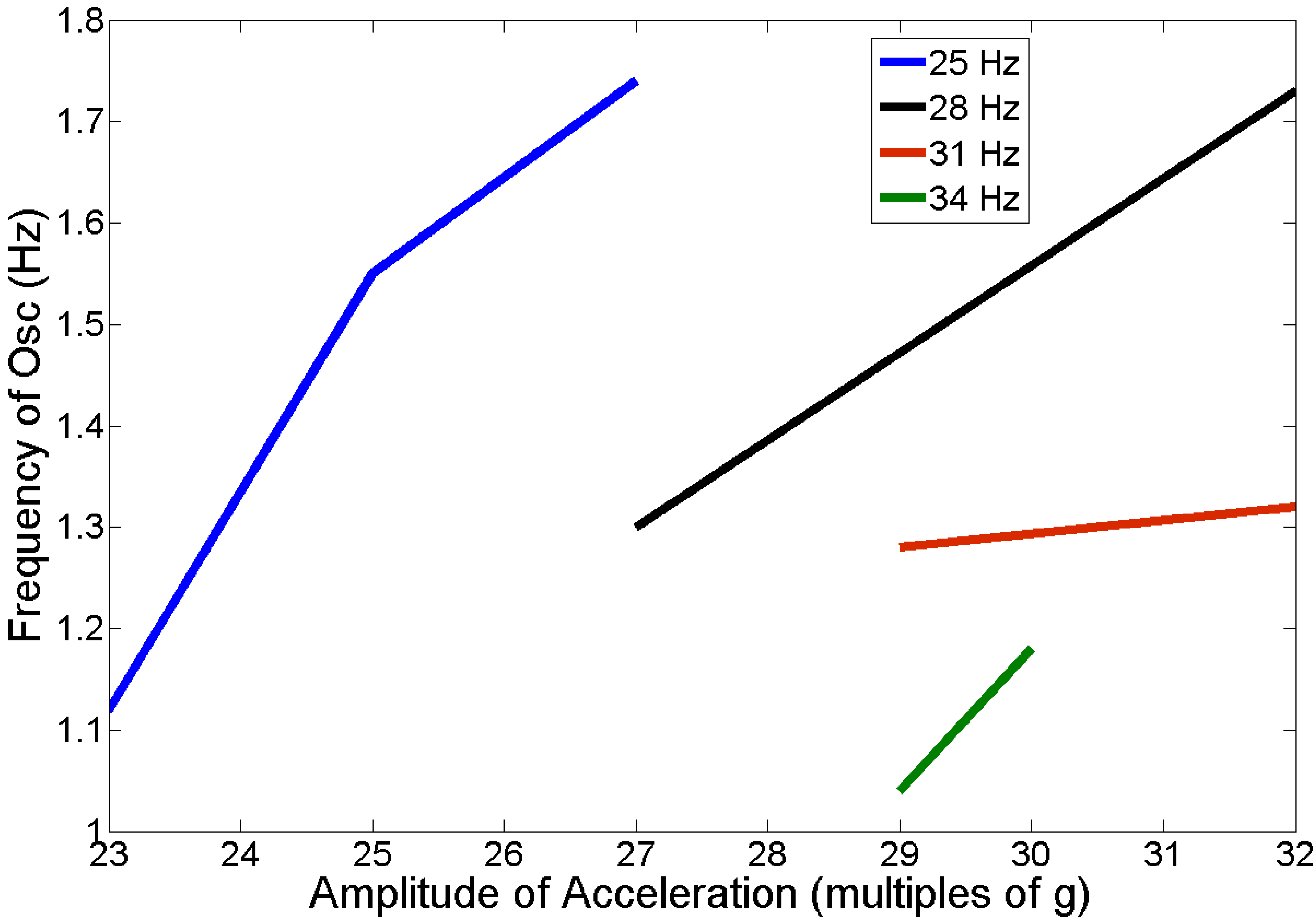
Stability Curve



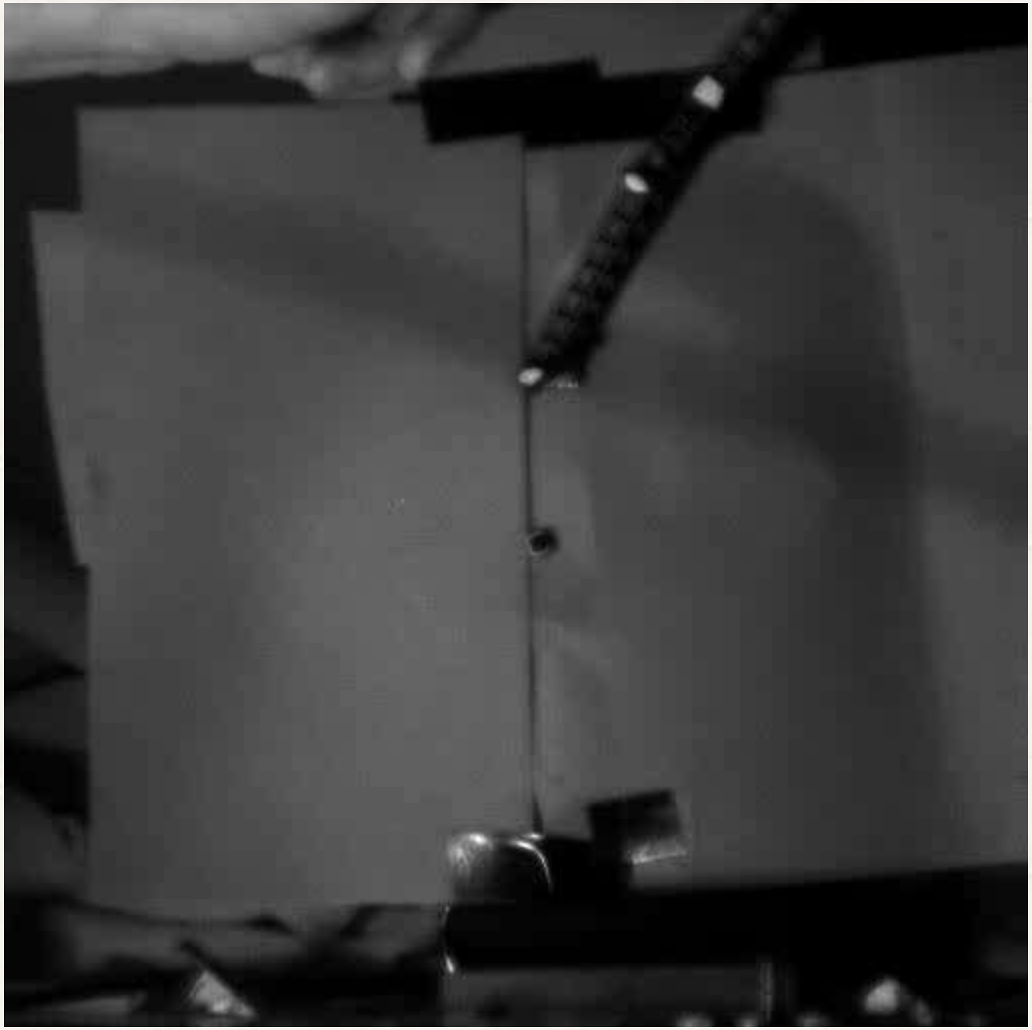
Small Oscillations *about the Inverted Position*



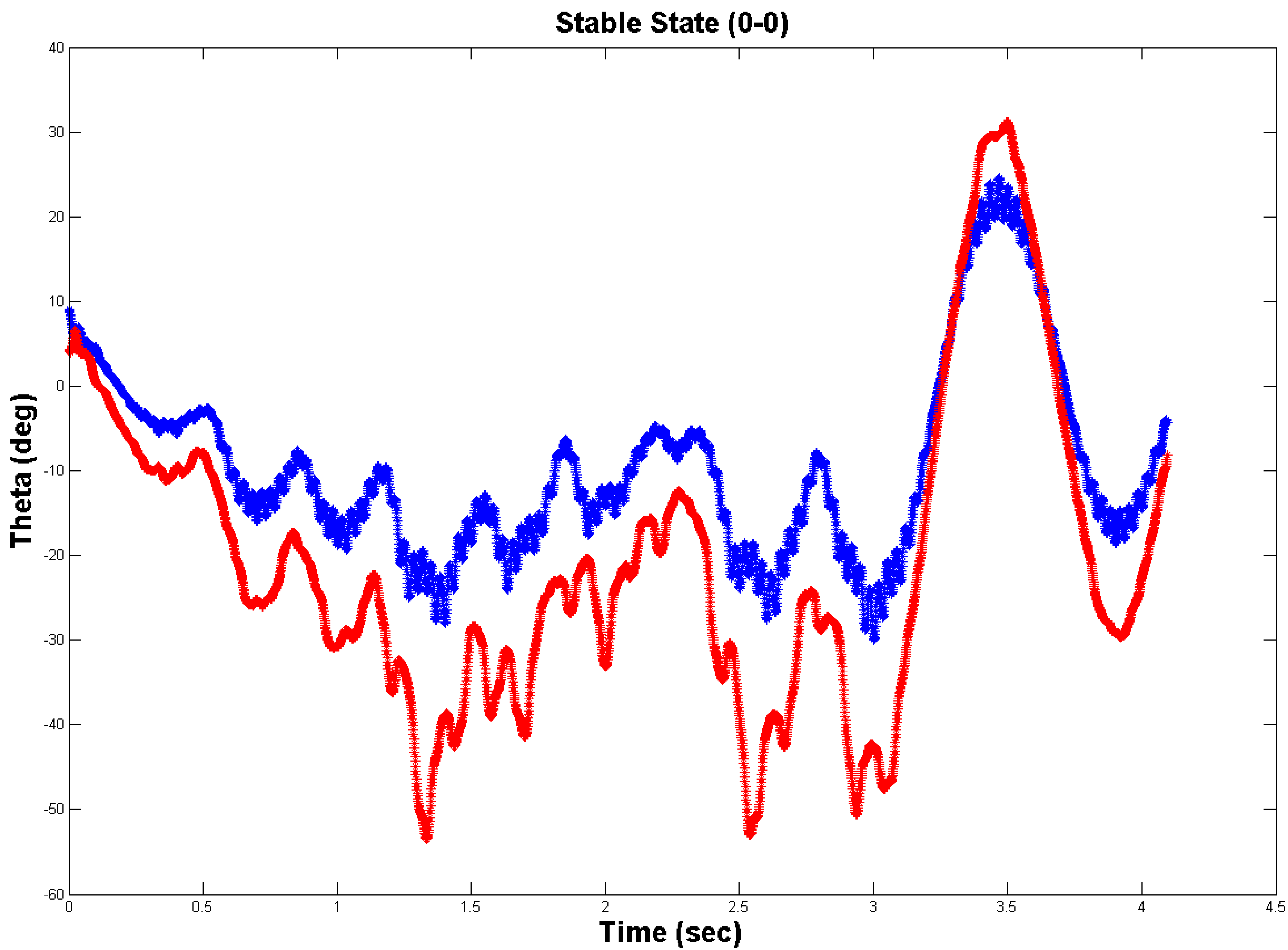
Small Oscillation Periods



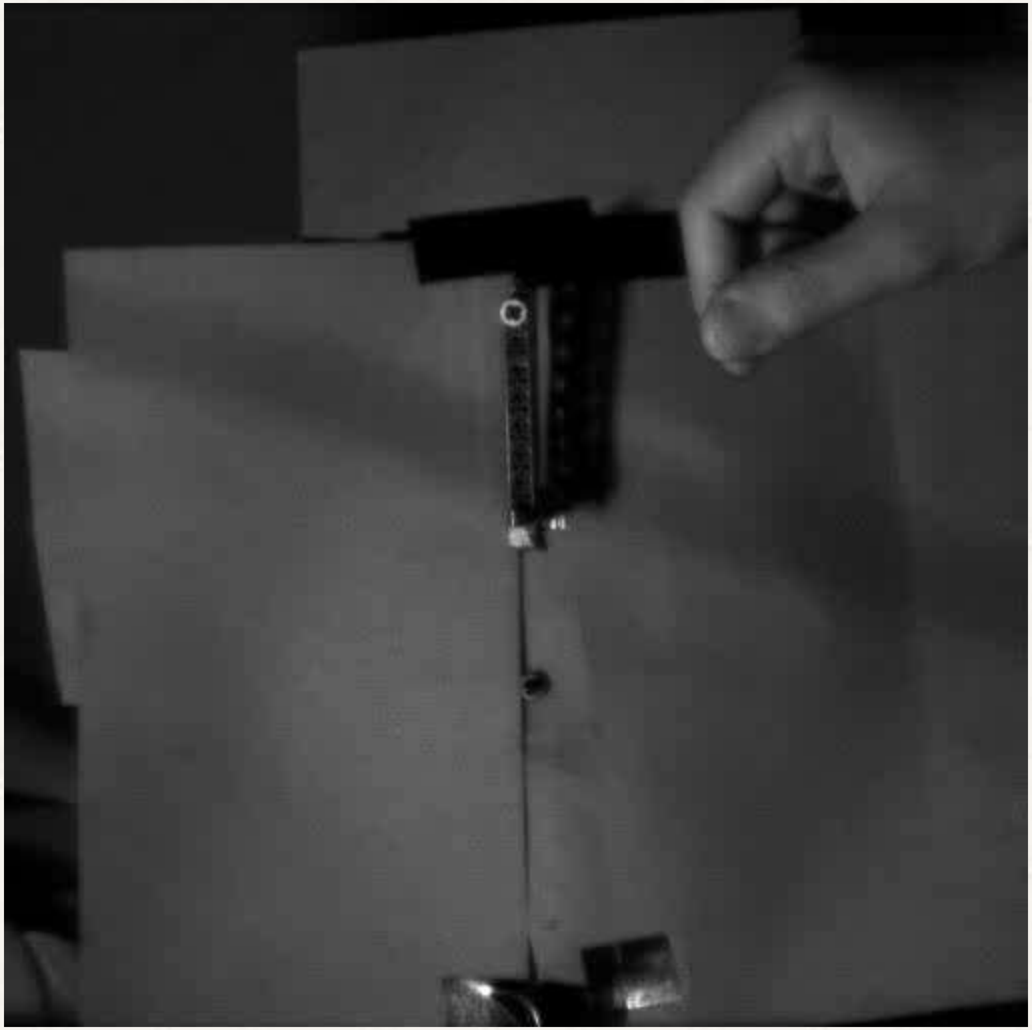
Double Pendulum Stabilization



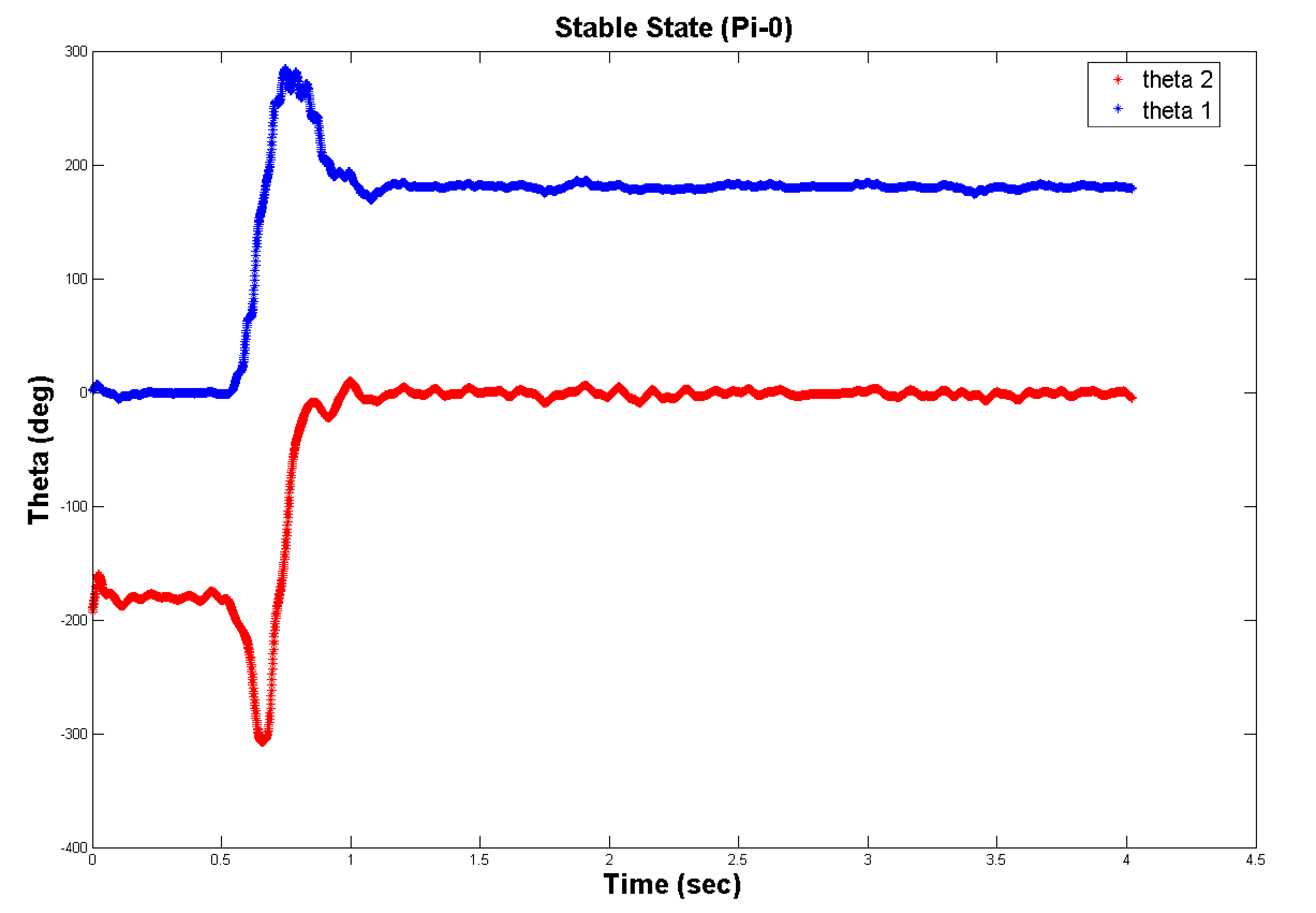
Double Pendulum Stabilization



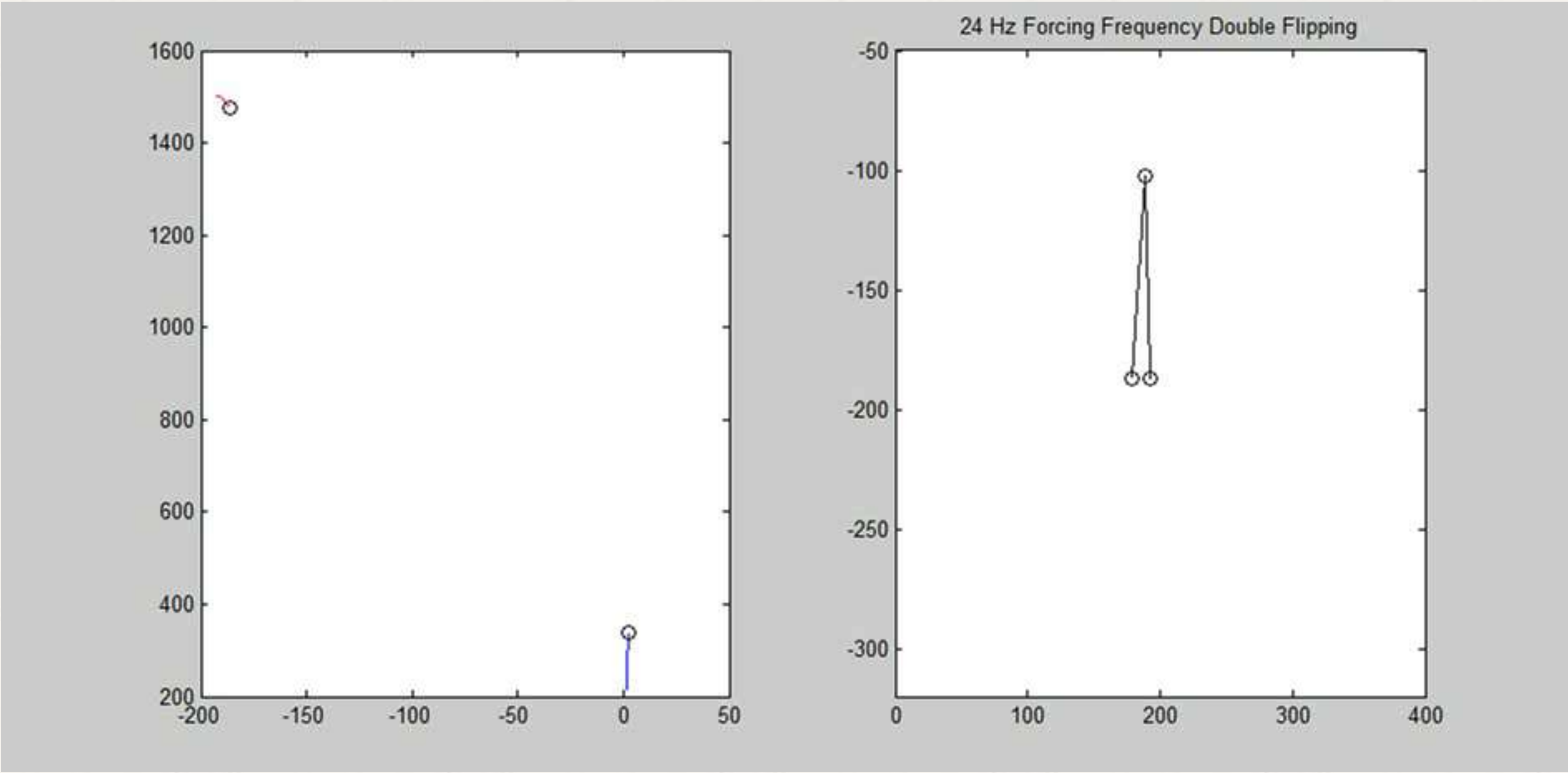
An Interesting Stable State



An Interesting Stable State



Phase Portrait Construction



Closing Remarks and Future Work

The Obvious

Inverted pendulum can be stabilized with only vertical, sinusoidal driving of the pivot.

Inverted *double* pendulum can also be stabilized by this method.

Both have nontrivial stability boundaries.

The Not So Obvious

Frictional[5] damping stabilizes the inverted state.

Double Pendulum exhibits separable behavior.

Despite idealizations and simplifications, modeling the system of ODE's exhibits the same qualitative dynamical behavior as the experimental data.

[5] Marchewka, A. , Abbott, D. , & Beichner, R. (2004). Oscillator Damped by a Constant-magnitude Friction Force. *American Journal of Physics*, *72*(4), 477-483.

Future Investigations

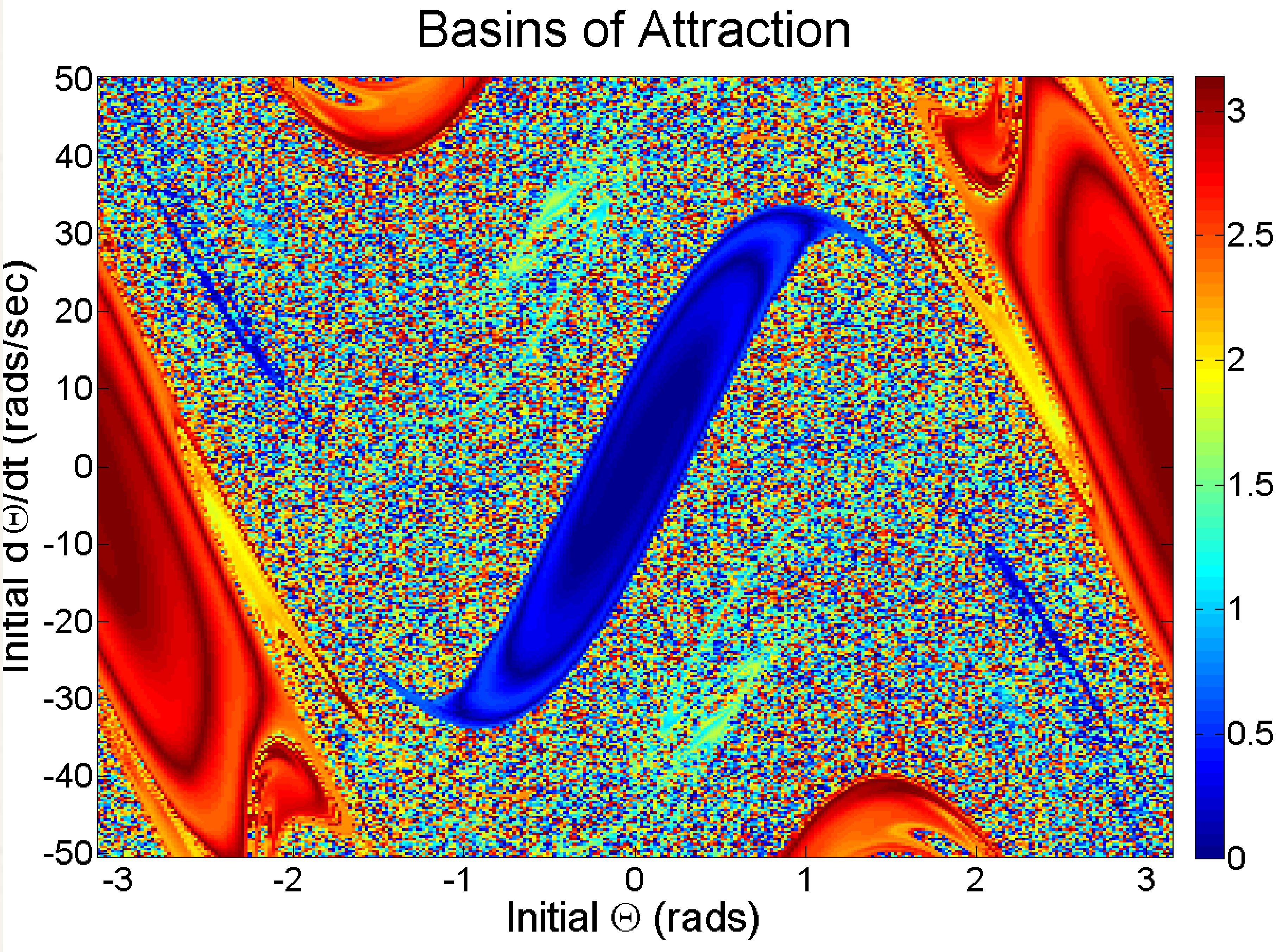
Period-doubling cascade (Mathieu equation)
Resurrection series[6], Chaos

Stability region analysis of the double pendulum

Evolution of Basins of Attraction as parameters vary

Three pendula? Four?
Why stop there: $N \to \infty$, continuum!

[6] P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953), Sec. 5.2; J. Mathews and R. L. Walker, Mathematical Methods of Physics (Benjamin, New York, 1965), Sec. 7.5.



Basins of Attraction
Initial dΘ/dt (rads/sec)
Initial Θ (rads)
50
40
30
20
10
0
-10
-20
-30
-40
-50
-3
-2
-1
0
1
2
3
0
0.5
1
1.5
2
2.5
3

Gratitude and Thanks

Nick Gravish

Dr. Daniel Goldman

Accommodating personnel in the Crab Lab