Drip Drip 'Til You Drop:
An Investigation of Dripping Faucet Dynamics

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In this paper, we investigate the dripping faucet as an example of a chaotic system. Using an improved mass-spring model, we simulate the dynamics qualitatively, reproducing all the major behaviors seen in experimental systems and the literature. Experimentally, we are able to reproduce one, two, and three-period motion, as well as chaos, closely matching the literature. Our major insights were in method; we were able to learn the best way to setup the experimental apparatus as well as the appropriate methods for data analysis.

I. INTRODUCTION

Ever since the publishing of "The Dripping Faucet as a Model Chaotic System" by R.S. Shaw in 1984, followed by a paper by Martien et. al. in 1985, the dripping faucet has been the subject of avid research[5,7]. Since those seminal works spawned the field a quarter century ago, numerous individuals have investigated its dynamics, both theoretically and experimentally[1,3,4,6,8]. What about a dripping faucet could spark so much work to be done on the subject?

A dripping faucet is an incredibly complex system if one wishes to model it precisely. It would require modeling the full Navier-Stokes equations in three dimensions, something which still can only be done numerically, and for which it is unknown if there is even an analytical solution (it is actually one of the seven Millenium Problems[2]). Shaw demonstrated experimentally that the dynamics of the dripping faucet are nonlinear, and in fact exhibit chaos, somewhat analogous to the logistic map[7,9]. The variable measured in these experiments is the period from the formation of each drop to the formation of the next. The system is attractive from an experimentalist's point of view because of its relative simplicity–all one needs, in principle, is a bucket, some water, a hose, and some means of measuring when drops occur. It is also attractive for its familiarity–it is an example of chaos in a system which is familiar to everyone, and yet seems, on the face of it, very simple. For these reasons, it has become a canonical example chaos in mechanical systems.

Our research project had two main aims: 1) observe the period doubling cascade to chaos seen in previous research, notably Dryer & Hickey[3] and 2) develop a computer simulation of the system.

In order to model the system, we need to know the equations of motion. Shaw originally proposed

$$\frac{d(mv)}{dt} = mg - ky - bv \qquad (1)$$

This models the system as a damped, driven harmonic oscillator. Here, y is the vertical position of the center of mass of the drop, v is the velocity of the drop vertically (it is assumed to have no other motion), k is the restoring force, b is a viscous damping, g is the acceleration due to gravity, and m is the mass[5,7]. This is a simple phenomenological model–the droplet is like a mass acted upon by gravity, a damping term coming from the water's viscosity, and a spring-like restoring force arising from the surface tension. Here, the flow of water is modeled by linearly increasing the mass of the drop in time. When the vertical position reaches a critical value, one resets m,v, and y to simulate droplet pinch-off, in a way that is dependent on conditions prior to pinch-off.

While this model is conceptually clear and excellent pedagogically, it suffers from several clear problems. The most obvious is the behavior of the spring constant k—it is a constant. However, during pinch-off, we know that the surface area connecting the drop to the rest of the water stream decreases, and so the restoring force should decrease. A better model would incorporate some dependence of the spring constant on the instantaneous mass of the droplet. Other problems are the model's ambiguity–Shaw provides no details on how precisely the resetting is done, nore what parameter values he uses. This works well for his case, as he was not attempting to provide anything but a qualitative similarity. However if one is trying to get some idea of the behavior of the actual system, a more precise formulation is desired.

Based on advanced hydrodynamical models, which solve a simplified version of the Navier-Stokes equations, Kiyono and Fuchikami in 1999 proposed a revised and improved mass-spring model[4]. It has the benefits of both providing explicit formulas for resetting the mass, velocity, and position at pinch-off, and also includes a linear dependence of the spring constant on the mass–a simplification, to be sure, but one which eliminates the problem of constant k to first-order. Their revised equations of motion are:

$$k(m) = \begin{cases} -11.4m + 52.5 & \text{if } m <= 4.61, \\ 0 & \text{if } m >= 4.61 \end{cases} \tag{2}$$

$$\dot{m} = Q = constant \tag{3}$$

$$m\ddot{z} + (\dot{z} - v_0)\dot{m} - kz - y\dot{z} + mg \tag{4}$$

$$m_{new} = 0.2m + 0.3 \tag{5}$$

$$z_{new} = z_0 = 2.0, \frac{dz}{dt} = 0 \tag{6}$$

The notation here is the same as before, replacing y with z and b with γ. v_0 is the velocity of the flow. The term $v_0\dot{m}$ is ignored as it is extremely small, just as in Kiyono and Fuchikami[4]. The units used in the simulation are such that g can be set to one: length is in units of 2.7 mm, time is measured in 17 ms intervals, and mass is measured in 20 mg. These units will be used from now on unless otherwise specified. This model exhibited similar qualitative behavior between the experimental results in such works as Dryer & Hickey[3], and also matched qualitatively with the behavior of Kiyono & Fuchikami's own hydrodynamical models[4]. This system of equations has the added benefit of being far more tractable computationally than the advanced hydrodynamical models.

From both the hydrodynamical models and previous experiments, we expect to be able to see a period-doubling cascade to chaos for increasing flow rates, and also a regime where we have a cascade from 3-period to chaos that exhibits hysteresis[3,4]. Thus, the dripping faucet should exhibit all of the major components of nonlinear and chaotic systems.

II. METHODS

A. Experiment

The experimental setup, as originally proposed, is modeled heavily on Somarikis et. al.[8]. A hose is attached to a feeder tank, which is maintained at a constant hydrostatic pressure by another reservoir tank. The drops are observed by a laser-photodiode system connected to an analog-to-digital converter (ADC); each time a drop falls, it passes between the laser and photodiode causing the laser to refract, cutting off the signal to the photodiode. This is then converted to a digital format and read in by a computer for data analysis.

Our first attempted setup was to use a plastic bucket with a hole drilled into the bottom of it, and a plastic tube secured in the hole by a hot-glue gun. This setup did not work very well. The biggest problem was controlling and measuring the flow rate–changes in water height were difficult to measure. Another major issue with this setup was variations in the direction of the drops. With the bucket, in order to get well-formed drops, it was necessary to use a small-diameter tube. The imperfections resulting from cutting of the tube to the appropriate length were magnified by the small size, resulting in the droplets breaking off with variable directions. Some of the droplets would miss the laser/photodiode sensor completely, resulting in very inaccurate measurements for the period between drops. This rendered data analysis effectively impossible.

Our second attempted setup was to abandon the idea of using hydrostatic pressure to regulate flow rate and instead use a syringe pump. This technique suffered from similar flaws as the previous. The syringe could only hold 60 mL of fluid, and so in order to form droplets in any regime other than single-period, we were forced to use very small diameter tubing in order to conserve water. If we were to use a bigger tube, the syringe would be emptied before the transients of the motion could die away. This was not the largest problem, however. The syringe pump exhibited a strong cycling behavior–while pumping, the rate of pumping would cycle, increasing and decreasing periodically as the motor turned. This could actually be seen by the naked eye, and was strong enough to cause a shift to a constant stream of water and back to individual droplets. This variability in flow rate would completely mask any of the behavior from droplet formation itself, rendering the syringe pump useless.

Our final setup employed a large pump/flow regulator to control the flow rate. This was, essentially, the same concept as the syringe pump, however the flow regulator did not appear to exhibit the cycling behavior. Since we were able to actually close the water's path of motion through the experiment, we were able to run the experiment at arbitrary flow rates and for arbitrary times, subect to the limits of the resolution of the flow regulator's controls. It was determined that the flow regulator's controls were in multiples of 0.004 mL/s. Thus, it was possible to control the flow rate down to an accuracy of 0.004 mL/s. This setup solved all the major problems. See Figures 1 and 2 for pictures of the setup.

The data from the photodiode and ADC is collected using a visual interface in LabVIEW created by TA Nick Gravish (see Figure 3 for example of the data as seen in the VI). It is then outputted in structured text files. Data analysis was performed in MATLAB on these text files.

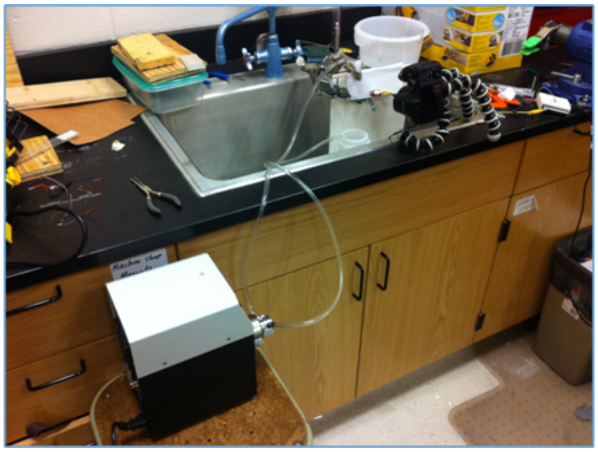

FIG. 1. Wide-view of the final experimental setup.

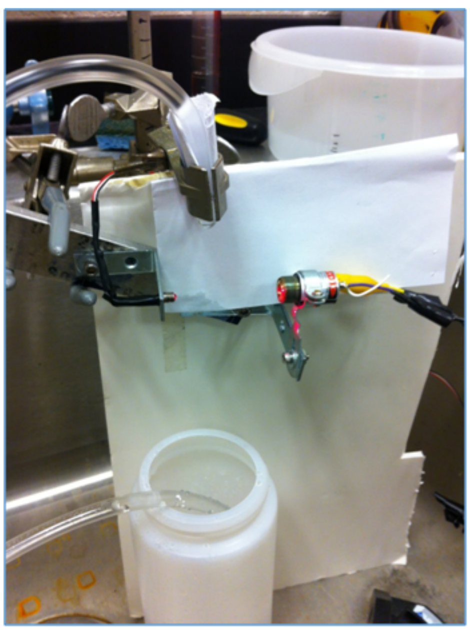

FIG. 2. Close view of the final experimental setup.

B. Model

The improved mass-spring model described in equations 2 through 6 was implemented in MATLAB using a fourth-order Runge-Kutta integration method that was written by the authors, rather than the built-in ODE45 differential equation solver. Using the MATLAB code, it was possible to output graphs for the time between drops, the motion of the drop as a function of time, the mass and spring constant as a function of time, produce plots of the period vs. drop count, and comparisons between the mass and period for the present drop vs. the next drop (i.e. Poincaré sections). The code is available for review upon request.

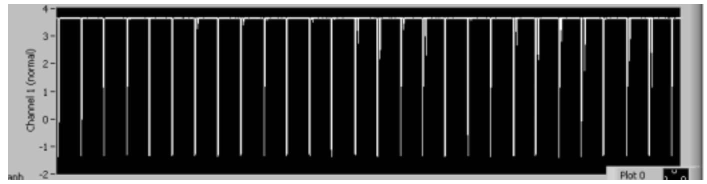

FIG. 3. Data output of the VI in LabVIEW

III. RESULTS

A. Data Analysis Technique

In order to extract meaningful data, there are two problems that must be solved. The first is that when a droplet pinches off, it forms a series of satellite drops, the number of which appears to be totally random. These satellite drops, if counted as "real" will radically alter the landscape of the T_n vs. T_{n+1} Poincaré section, or return map. It was not until well into our experiment that we realized this—we had interpreted these to be what was meant for one, two, four period,

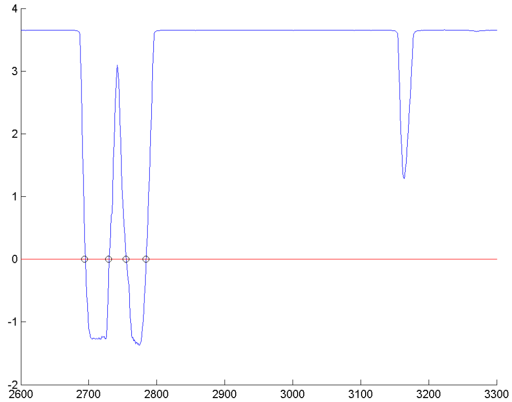

FIG. 4. Blown up view of the data from a single drop for purposes of visualization of data analysis technique: Redline is the threshold, dots represent crossings of the threshold.

etc., which dramatically effected our interpretations. In order to correct for this and be able to extract the true periodicity, a threshold value was implemented for the signal from the photodiode—only when the signal from the diode drops below this threshold will the code report a drop. The satellite drops are far smaller than true droplets, and so an appropriately assigned threshold will eliminate all satellite drops yet retain all real droplets.

The other phenomenon that had to be corrected for was "debouncing." This is where we see, in each "spike" of the signal which signifies a droplet passing in front of the laser, a double-peak (see Figure 4 for a visual of the data analysis process). That is, that instead of a single well-defined turning point where the refracton of laser light reaches a maximum, we see two peaks seperated by a valley. This is caused by the spherical shape of the water droplet–when the droplet first passes in front of the laser, the laser travels through a highly curved region and so is strongly refracted; when the laser is passing through the center of the drop it is refracted far less; and when the drop has almost left the path of the laser the laser is again strongly refracted. In order to remove this, we wrote our data analysis codes such that only the first crossing of the threshold is counted, not the second. So, when a droplet first intersects the laser's path, it registers as a drop; when the droplet is leaving the laser's path, and so the signal passes back under the threshold value, we ignore it. We also implemented hysteresis, i.e. a minimum time, in order to prevent particularly strong debouncing behaviors from being recorded as two seperate drops (see Figure 4 for a visualization of this). Together, these methods allow us to record the passing of a droplet through the laser only once, and record only the passes of real droplets.

In order to visualize the data, plots of the period of the present drop (T_n) vs. the period of the next drop (T_{n+1}) were produced, as well as plots of the period of the drop versus the number of drops up to and including the given drop (which we called "drop number").

B. Results

The results for the experiment are displayed in Figures 5-9. These are either a Poincaré map of T_n vs. T_{n+1} or a plot of T_n vs. drop number. The green line in all plots is the line of symmetry. Each dot represents a single drop. A tightly-packed cluster of dots on the Poincaré map would represent a cluster of drops all of the same period, therefore the number of clusters is equal to the periodicity at the given flow rate. An alternative method of visualization is found in the plots of period versus drop count. Here, a horizontal band of dots would represent drops of all the same period, so the number of bands is equal to the periodicity. Based on this, it is clear that we observed period-one and period-two behavior, as well as chaos at several different flow rates. As one can see, for a flow rate of approximately 0.210 mL/s, we see period-one behavior with a period of roughly six seconds (Figure 5). At 0.319 mL/s, we see period-two behavior with periods of roughly four and 4.5 seconds (Figure 6). Finally, as flow rates of 0.579, 0.374, and 0.456 mL/s we see the emergence of chaotic attractors (Figure 7).

One can see that all three of the chaotic attractors in Figure 7 look quite similar, at least qualitatively. They have a similar "M" shape on the left, and a straight tail on the right; this pattern holds even though these are taken at widely varying flow rates.

We were also able to find evidence of a 3-period regime at 0.365 mL/s (Figure 8). The Poincaré map, however, is not as "clean" as the others–there appears to be significantly more noise both between the clusters and the clusters themselves look larger. If one looks at the period versus drop count plot, one can see that there appears to be regions where there are three distinct bands, punctuated by regions of apparently random distribution of periods. This may be evidence of the transition to chaos from three-period and hysteresis of the system (that is, it behaves differently when increasing versus decreasing the flow rate) found in Dryer & Hickey[3].

Our model also produces one, two, and three period behavior, and these plots look essentially identical (aside from the scale of the axes) to the experimental diagrams, and so will not be reproduced here. The two most interesting structures of the model are its chaotic attractor (Figure 9) and the bifurcation diagram for varying flow rate (Figure 10). These will be discussed in more detail below, however let us simply note that the chaotic attractor looks broadly similar to the experimental attractors, and that the bifurcation diagram of period vs. flow rate exhibits period-doubling cascades to chaos, as well as regions of three-period emerging from

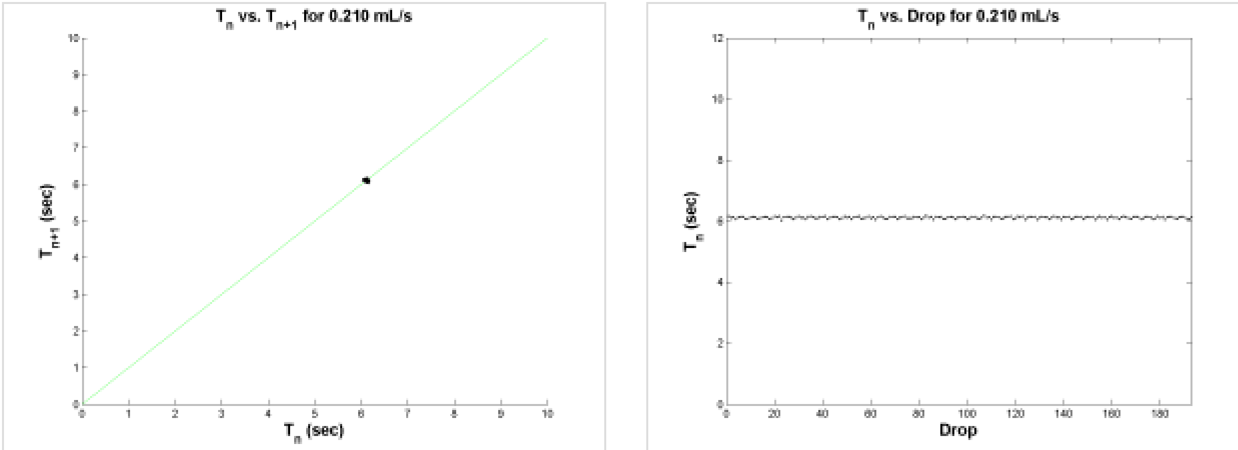

FIG. 5. Period one data, at 0.210 mL/s. Poincaré map of T_n vs. T_{n+1} (left) and plot of period versus drop number (right)

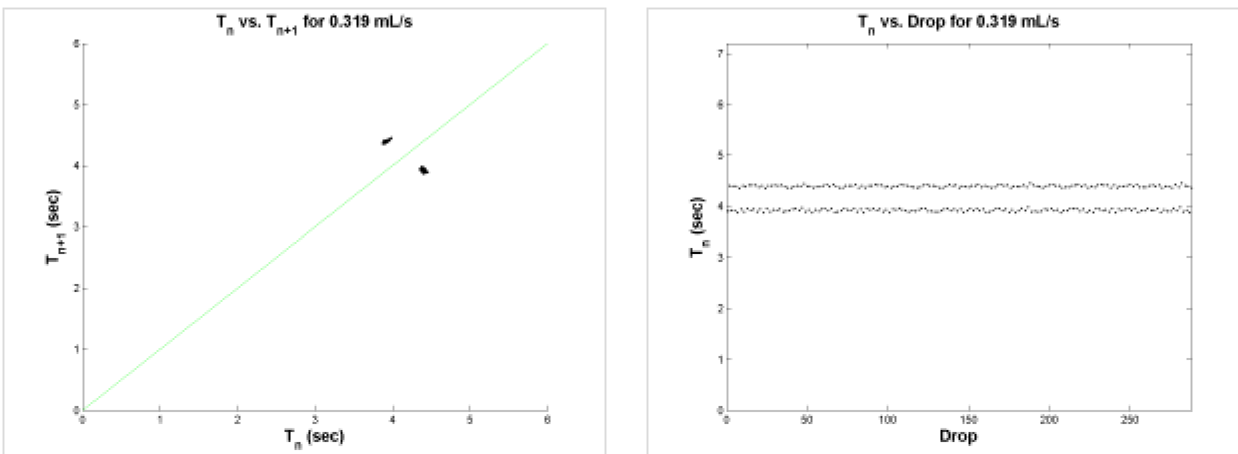

FIG. 6. Period two data, at 0.319 mL/s. Poincaré map of T_n vs. T_{n+1} (left) and plot of period versus drop number (right)

chaos.

IV. DISCUSSION

Our experimental results match qualitatively well with the results found in the literature, most particularly Dryer & Hickey[?] . We clearly achieved period-one, period-two, and chaotic behavior. Figure 8 shows what looks to be a three-period Poincaré map combined with that of a chaotic regime. This is also seen in its period versus drop number plot, suggesting that at 0.365 mL/s is right on the edge between the chaotic regime and a three-period. We conclude that we have actually observed the three-period route to

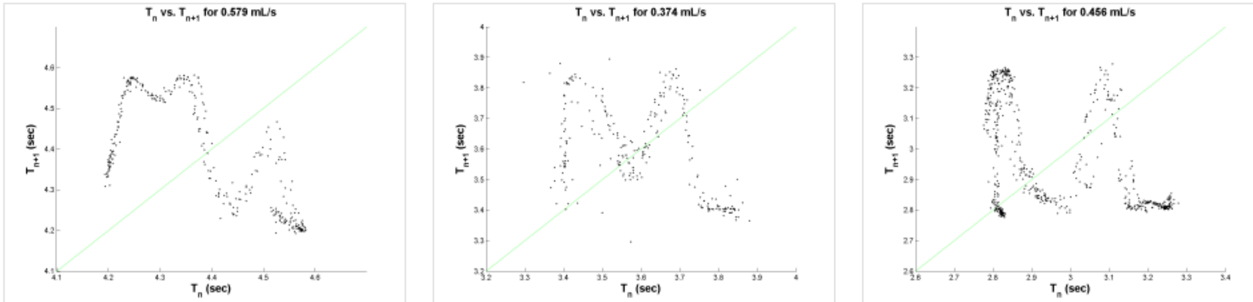

FIG. 7. Chaotic data. Poincaré map of T_n vs. T_{n+1}. Flow rates are 0.579 mL/s (left), 0.374 mL/s (center), 0.456 mL/s (right)

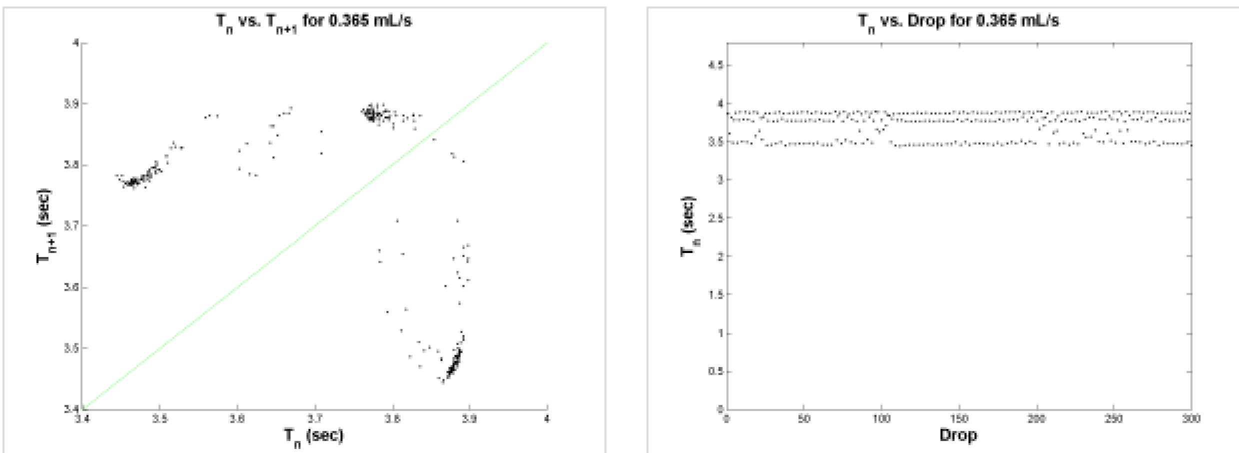

FIG. 8. Period three data, at 0.319 mL/s. Poincaré map of T_n vs. T_{n+1} (left) and plot of period versus drop number (right)

chaos.

One question that might be asked is where four, eight, and other periods are located, and why we did not observe them if we predict them theoretically. Our explanation relies, in part, on a blown-up version of Figure 5, the one-period regime. In Figure 11, we see that when we expand that cluster of points in Figure 5 that we actually see a circular motion of the period, with the period varying by roughly 2%. It is clear that this is a "one-period" region. Our means of understanding this is that the pump exhibits a similar, but far less pronounced, cycling behavior as we saw with the syringe pump. This would explain the very small shifts of the period that we see in Figure 11. We also know that the transition from two-period to chaos occurs in only a few multiples of the finest resolution of the flow regulator used in this experiment. When we combine this cycling behavior with the low resolution of the flow regulator (by

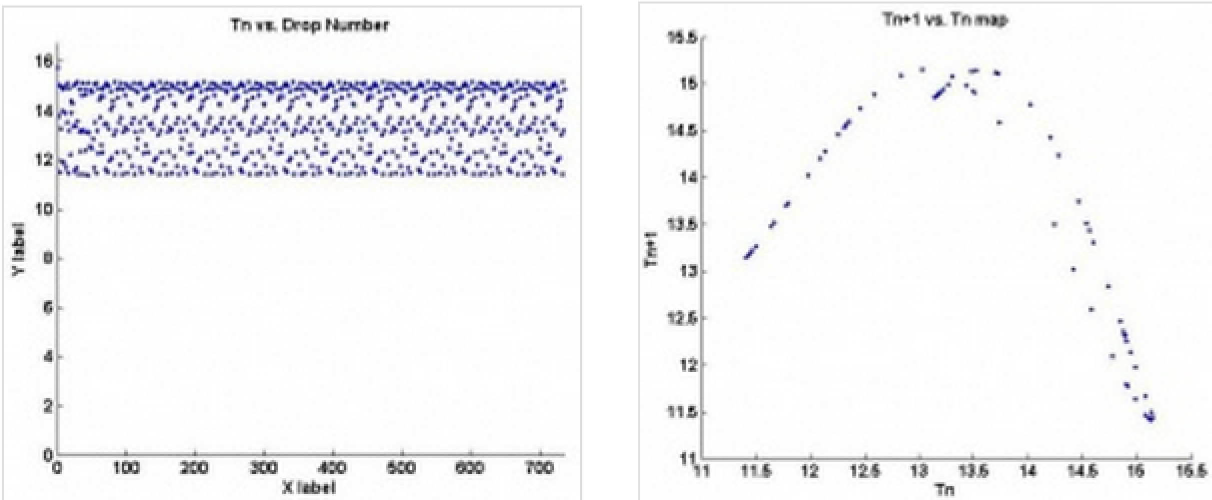

FIG. 9. Simulation in the chaotic regime ($v_0 = 0.115$). Plot of period versus drop number (left) and Poincaré map of T_n vs. T_{n+1} (right)

comparison to the size of the interesting region), we conclude that our system simple does not have the resolution and accuracy necessary to produce a stable period higher than two.

Our model produces qualitatively similar behaviors as are seen by experiment. Figure 10 shows the bifurcation diagram of the period as a function of flow rate. The period is trending downward as time goes on, but undergoes bifurcations to chaotic regimes, which then collapse back to single-period in a cyclic pattern as the flow-rate increases. This behavior is explained by the nature of the chaotic attractor–one can see in Figure 9 that it intersects itself, something that the experimental results and other common attractors (for instance, the logistic map) do not do[9]. This means that it is not what is called a "unimodal" map, which exempts it from some of the general theorems concerning chaotic systems described in Strogatz[9]. In particular, it means that the system can move beyond chaos back into a single-period again. This behavior is actually seen in the physical system as well, as seen in Dryer & Hickey[3], but our experimental flow-rate resolution was too low.

One final note should be made about the model: it is not, strictly, chaotic. Figure 9 actually plots several hundred drops, and yet there are precisely sixty-six dots there. The system is actually at a sixty-six period at a flow rate of $v_0 = 0.115$, as determined by examination in MATLAB. It also exhibits an extremely sensitive dependence on flow rate, a change by less than one percent in either direction ($v_0 = 0.1149 or 0.1151$) will cause the

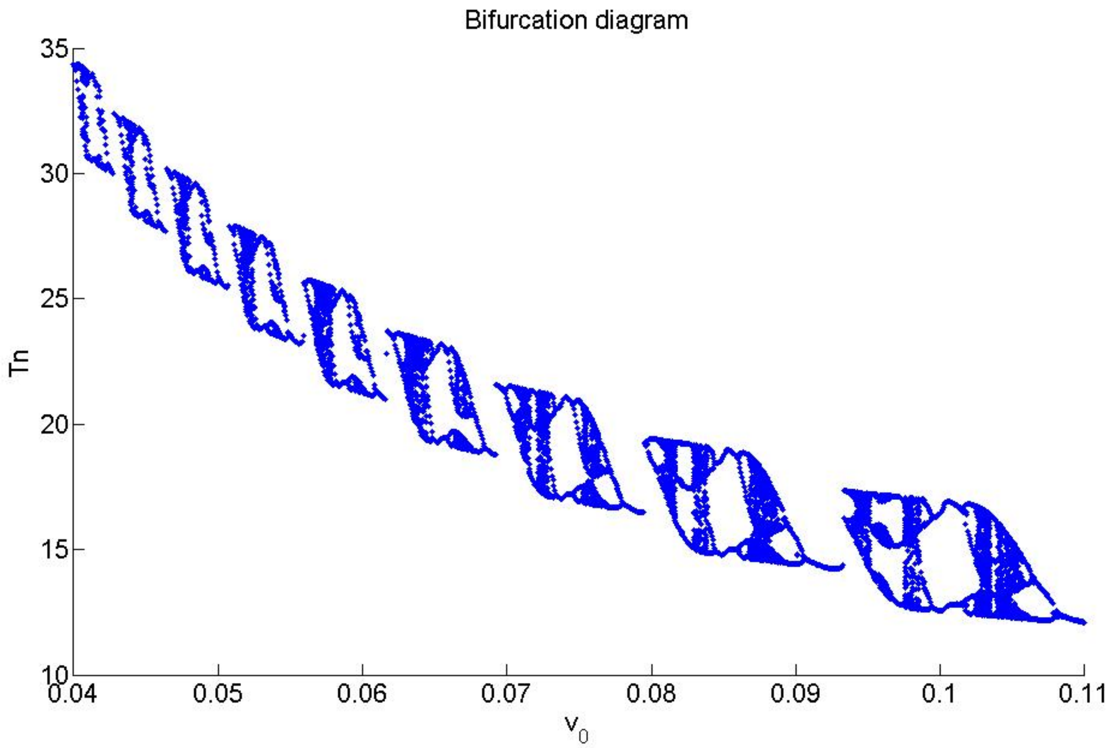

FIG. 10. Bifurcation diagram from simulation, flow rate varying from $v_0 = 0.04$ to $v_0 = 0.11$ on x-axis, period of drops on y-axis.

flow rate to fall to either twelve of fourteen period, depending on if the rate is decreased or increased. In any system exhibiting the type of slow variations in flow rate exhibited either by a draining bucket or by a system such as the experiment's pump regulator, this implies that the periodicity of the dynamics will be shifting more rapidly than one full period can complete. This will have the effect of making the dynamics actually appear to be truly chaotic, even if they are not. There is no reason to believe that this is what is occuring in the actual physical system, but it is an interesting thing to consider nevertheless.

V. CONCLUSION

The results of our model and experiment closely match their counterparts in the literature. Our model is able to reproduce all the results found in Kiyono & Fuchikami's paper[4], and produces similar qualitative results as seen in experiment–a cascade to chaos, both from single-period through two, four, and so on periods, as well as transitions between chaos and three-periods. The major new piece of information from our model is that it is, despite the claims of Kiyono & Fuchikami, not chaotic (in the strict sense of aperiodic)[4].

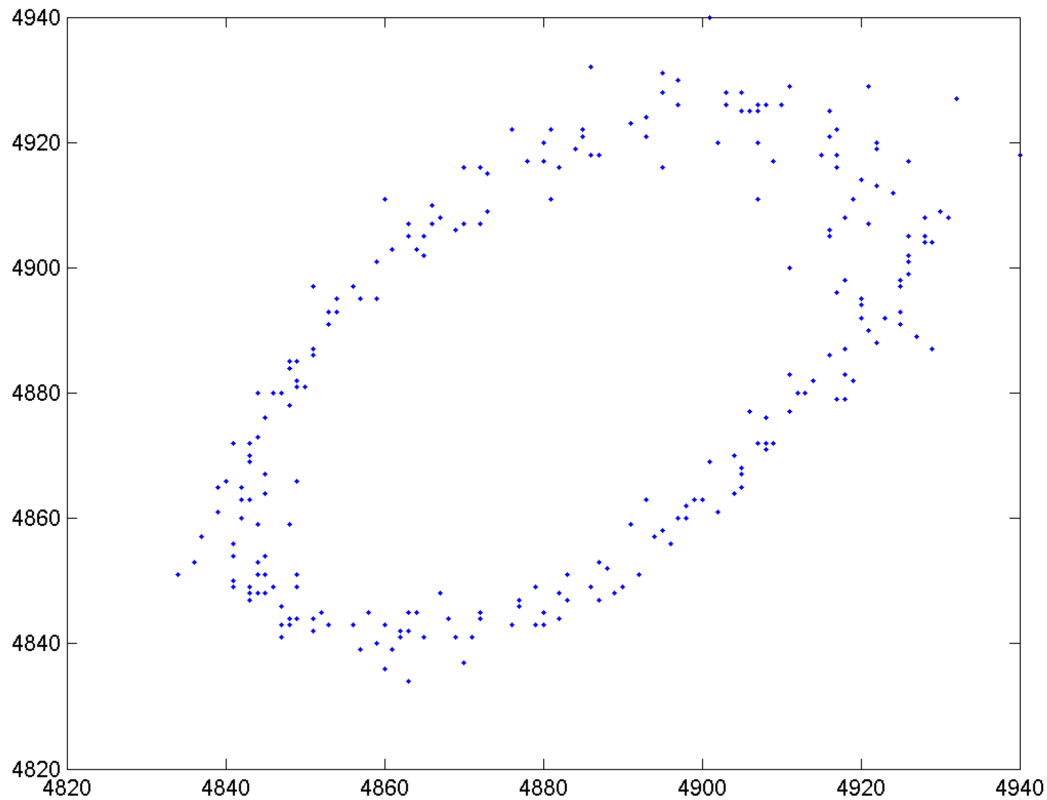

FIG. 11. Blow up of Figure 5, single-period motion exhibiting very small periodic changes in flow rate from pump regulator.

Experimentally, we were able to observe the transition from one to two-period motion to chaos, and saw evidence of a transition between chaos and three-period motion. We ascribe our inability to see higher-period motion to both the cycling behavior of the pump and the low resolution of the flow regulator. The major insights from the experimental portion of our work lie in the realm of method. We found that controlling the flow rate is very difficult when one is using hydrostatic pressure as the flow regulator, and that the best method for being able to closely control the flow rate is to use some type of pump system. We also found that tubes of a wider diameter perform better, with less noise, than smaller diameter tubing. Knowing this would be extremely helpful for any other experimenters seeking to perform this experiment in the future.

Future work in this area would occur on two fronts, both the simulation and the experimental. On the simulation front, it would be quite useful to be able to implement the more computationally intensive hydrodynamical models as laid out by Kiyono & Fuchikami[4] and Coullet[1] et. al., in order to better simulate the real dynamics. The major new effort experimentally would utilize a proposal from Dr. Dan Goldman. In order to improve resolution, we would shunt a small portion of the flow coming from the flow regulator, thereby reducing the effective flow rate. This would increase the effective resolution of our control on the flow rate, allowing us to investigate more thoroughly the period-doubling cascade to chaos.

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