EPJ manuscript No.
(will be inserted by the editor)

Considerations for using Match Stick Experiments to Study Wildfire Dynamics

Benjamin Musci[1], Lehzeng Fang[1], and Mateo Reynoso[1]

Georgia Institute of Technology

Received: 12/11/2020

Abstract. This article attempts to asses the efficacy of using desktop experiments using matches to study the dynamics of wildfires. Experiments were conducted using 1D arrays of matches and a simple flame tracking algorithm. The effects of parameters (match spacing and inclination angle) on flame spread rate were investigated. Using these results general comments can be made about fire spread dynamics. This also allows one to asses some concerns when using matches to study wildfire dynamics.

1 Introduction

The importance of understanding wildfire dynamics hardly needs defended at this current moment. As one of the worst fire seasons in United States history finally comes to an end, many areas have been left ravaged. Additionally, just one year ago Australia's nightmare 2019 fire season drew global attention. As the climate continues to warm and the prevalence of drought increases in the coming decades, the occurrence and severity of wildfires across the globe will continue to be a problem. Thus a better understanding of wild fire dynamics will help modeling and predictive efforts to try and mitigate the societal damages these natural disasters create.

While there may be no question about the need for a better understanding of wild fire dynamics, there is a large deal of uncertainty in how they should be modeled. The parameters most related to the spread of wildfires are still not understood; from those intrinsic to the forest or fire itself to the conditions created by weather or past forest management practices [1]. For example, the ability of current models to recreate and predict fire spread is poor. Widely used empirical models are prone to under-prediction, while more physically based models are much more expensive and have not been robustly compared to real data-sets of requisite resolution. [2] Not only is it a question of identifying the most pertinent parameters, but also finding how they can most efficiently and effectively be incorporated into a model of a system so complex as the 3D coupled nonlinear behavior of a wildfire. The range of spatial and temporal scales over which these processes occur complicate the problem further [3].

First it is important to specify which type of wildfire that needs modeled. The type of fire typically associated with the events that burn hundred of thousands of acres are crown fires. In crown fires the fire has engulfed the fuel at the surface level and consumes the fuel at the canopy. These large vertical flames then propagate from tree crown to crown through the forest. While it requires much more energy to start a crown fire (due to the added moisture in the foliage at the canopy) these fires are much more destructive and have higher rates of spread and ember transport compared to surface fires [2,3]. Therefore, this work will focus on relating table top experiments to crown fire behavior. The fire characteristics of the experiments will then be analyzed and compared, from a qualitative point of view to the dynamics of fire models and data.

Further, because the experiments conducted used only 1D arrays, the relation to actual fire dynamics is limited. However, qualitative dependence on key parameters and characteristic time scales can still be identified and discussed. This information can be further used in assessing the limitations and merits of using larger 2D match stick arrays (such as those in the work of [5]) to study fire dynamics.

2 Methodology

2.1 Experimental Setup

This section covers the experimental set up and procedure. In order to reliably control the inclination angle and spacing between matches, pieces of foam board were used as the base for each "run" or individual experiment. This allowed

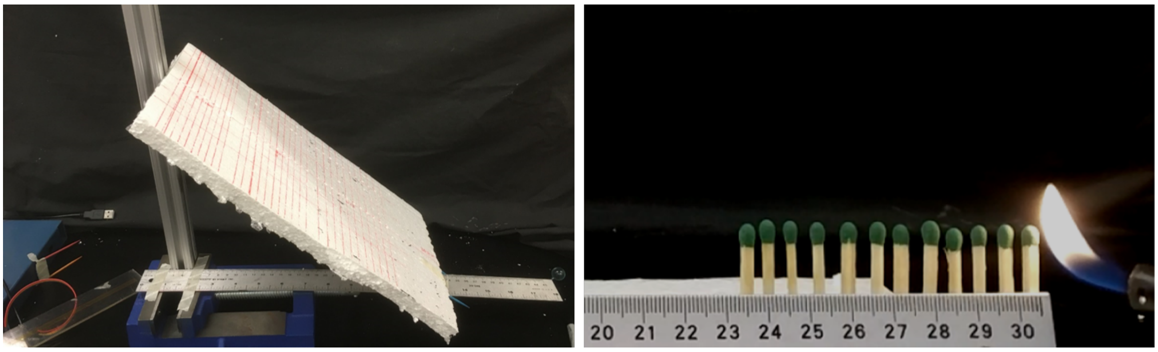


Fig. 1. The experimental setup. Left) An example of an angled run. Matches were inserted into the foam at marked increments. Right) An array of matches about to be lit. The ruler was used in each image for calibration

one to mark the board at a certain increment and ensure the base of each match was placed exactly at that spot. The mach spacing parameter is intended to be used as a proxy for forest density in wildfire studies. However, having the base of the match in the foam board did not ensure the match heads were the correct, or even equal, spacing apart. One can see this in the right side of Figure 1 as some match heads are closer than others. This introduced experimental error that can be observed in some of the results. Regardless, effort was made to ensure somewhat equal spacing at the match heads for each run.

The use of the foam board also allowed for the variation of the surface inclination angle. This parameter was used to simulate the topography in forests and an example set up is shown on the left of Figure 1. These two parameters, inclination angle and match spacing are the two variables used to explore the dynamics of 1D flame spread in this study.

Once the match spacing and inclination angle were set, a ruler was placed in the plane of the matches in order to have a reference for calibration in post-processing. A grill lighter was then used to light one end of the array, and a cell phone was used to record video. Once all matches in the array were lit and the flame stopped propagating across the array the video was stopped.

2.2 Data Processing

Once the videos were exported off of the cell phone, they were converted to an image sequence and analyzed frame by frame using a combination of ImageJ, Matlab, and Python. Of primary interest in this study was the rate of flame spread for different parameter combinations. In essence this was acquired by first extracting flame front position, filtering, and then taking the first derivative to find an average flame speed.

The step by step process was more complicated due to the flame dynamics. Each image is thresholded to a binary image. The centroid of the thresholded flame is then computed and it's location stored in an array. This is repeated for each image and the centroid position array is used as a proxy for the leading flame front position. This method was verified by using a manual tracking algorithm in ImageJ. The user selected the location of the leading flame edge in each frame in the entire image sequence. The resulting slope of this array was shown to agree well with the slope of the centroid flame position.

Because of the noisy nature of the flame dynamics, the flame position must be filtered before finding the average flame speed. As seen in Figure 2 the flame position is noisy, oscillatory, and can even be step-wise in nature. Thus a low pass filter was applied to this data over the region of interest (in order to omit the initial lighting and end periods), to acquire a smoothed position array. This smoothed array was then used to find a flame velocity for each run. The Fourier transform of the position array was also computed to find the dominant oscillatory frequency in the flames propagation.

3 Results

Of immediate note when conducting these experiments was the nonlinear behavior of the flame. As it propagates the flame whips about back and forth. This can be seen in the top of Figure 3. It also soon became obvious that this was in large part due to the oxidizing agent on the match tip. When the oxidizing agent ignites it explosively releases heat. This explosive serves to convect the ignited flame to neighboring matches, which in turn explosively ignite and continue on in this way. This process occurs on relatively short time scales and can lead to very rapid propagation. It can be seen in more detail in Figure 4. As the match ignites the heat release carries the flame over 2-3 neighboring

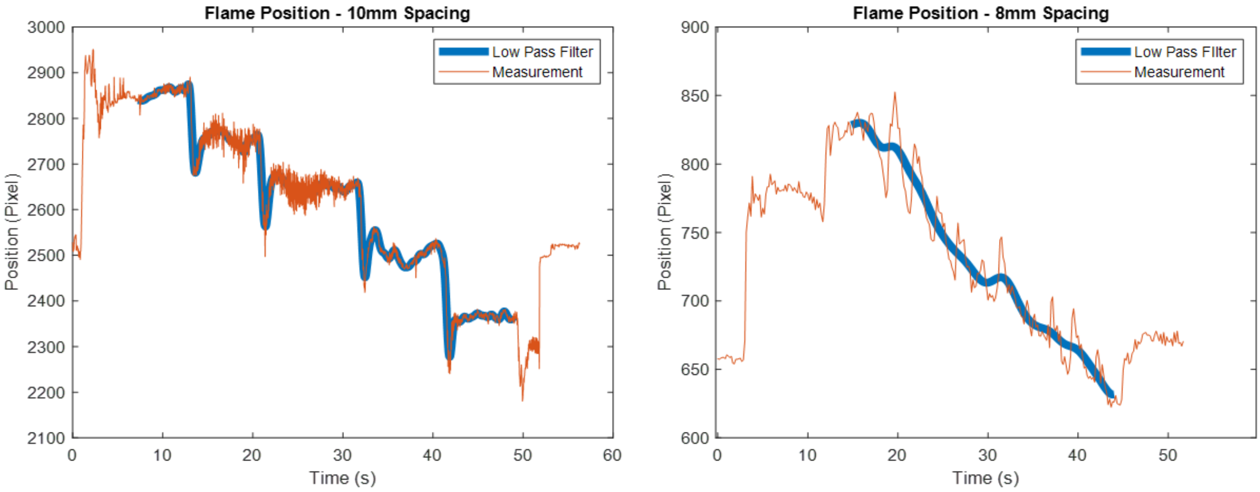


Fig. 2. Two examples of the resulting flame front position array. The orange curves are raw data, while the blue curves are low pass filters applied to the regions of interest. The left image shows a flame which propagates in a step-wise manner. One match lights, and then slowly heats the neighboring match for a few seconds before it ignites. This phenomena occurs near the critical match spacing. The right image shows a more typical array where the flame is continually propagating from match to match. Note the oscillatory behaviour of the flame in the data.

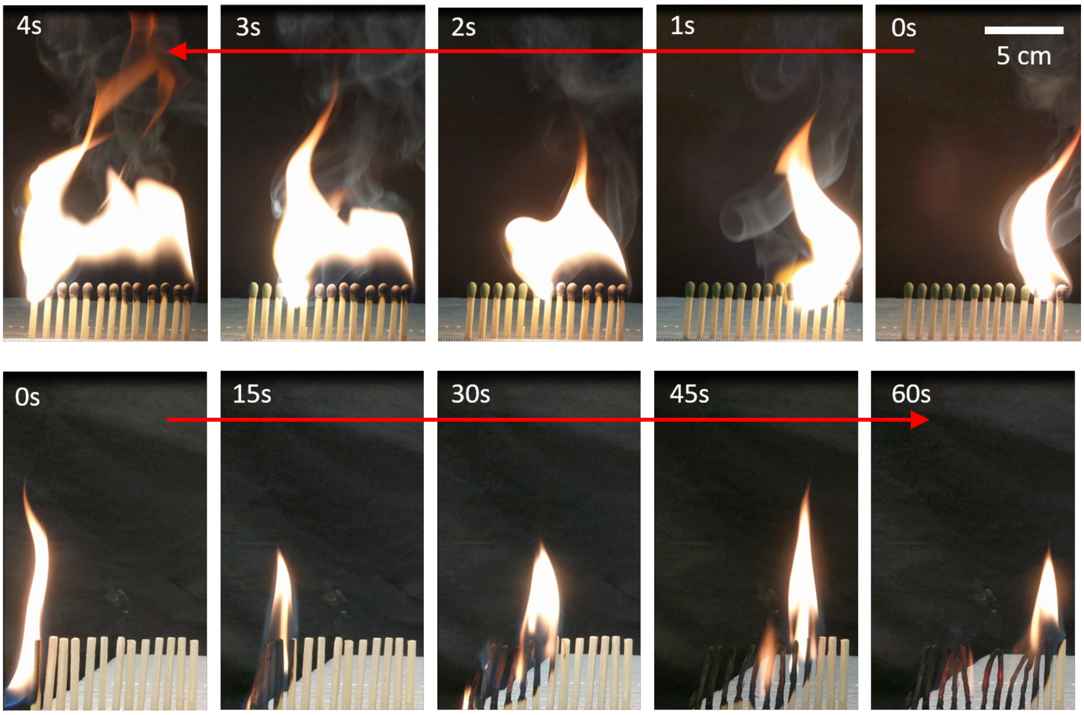


Fig. 3. Two run examples. The top sequence of still images shows flame propagation from flat matches spaced 5 mm apart, propagating right to left. The bottom sequence shows the same parameter combination, but with the match heads facing down. The flame propagates from left to right. Notice the difference in flame dynamics between the two cases. This highlights different modes of flame spread, and shows the effect of the match heads.

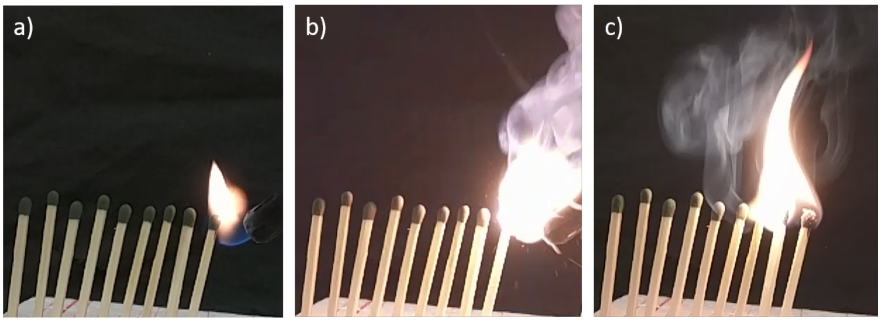


Fig. 4. The exploding action of the match tip. a) The match tip prior to ignition from the lighter. b) the oxidizing agent of the match tip igniting and explosively releasing heat. c) The explosive nature of the ignition carrying the flame over the two neighboring matches.

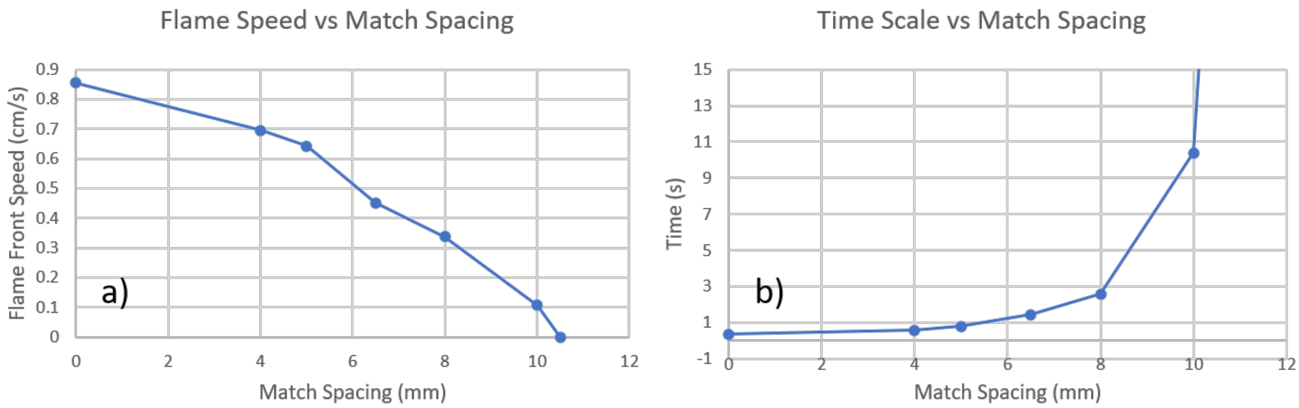


Fig. 5. Speed and time scale dependence on match spacing. a) The average flame speed form an ensemble of runs conducted at each spacing. b) The time scale (speed/match distance) for the propagation mode for same ensemble of runs.

matches, and then whips about as it travels down the line. This fast propagating method is thought of as a convective mode of heat transfer as the ignition advects the flame to neighboring matches. This mode dominates when matches are closely spaced. It's effect can also be seen in the right hand image of Figure 2 in the regularly spaced oscillation going down the slope of the array.

Another mode of flame propagation or heat transfer was also identified. When match spacing neared the critical distance, i.e when the distance was too large to allow for fire spread, the convective mode no longer dominated. With larger spacing a match would tend to ignite, whip about momentarily due to the oxidizing agent, and then burn calmly on its own for several seconds, in the meantime heating up neighboring matches primarily through radiation. Once the neighboring match reached some critical temperature it would ignite without the initial flame having ever physically touched it. This much slower mode of flame propagation is radiation dominated and occurs on a much longer time scale. It dominates at larger match spacing. The effects of a flame spread in this regime can be seen in the left image of Figure 2. The step-wise progression of the flame is caused by this mode as the flame front suddenly jumps forward, then sits for some time, and then jumps again once it's flame has radiated enough heat to the neighboring match.

Unsurprisingly, the outlined flame spread modes lead to a distinct dependence of flame speed on match spacing. This can be seen clearly in Figure 5 as the flame speed non-linearly decreases with match spacing. Once the spacing reaches a critical distance, in this case just above 10 mm, the flame front will no longer propagate. In some way this could be thought of a bifurcation, as the control parameter (spacing) is varied and goes below the critical distance a new solution is created that allows for flame propagation. Note that the 0 mm spacing case was for an array of match heads with each touching the other, representing the upper limit for flame speed in the flat case.

One can also find the characteristic time scale, τ, governing the flame spread. This was found by simply taking the match spacing divided by the average flame speed. The resulting time is a sort of match propagation time, or the time it takes for the flame to spread from match to match. It should be noted that the dominant frequency found by taking the FFT of the flame position array, roughly matches $1/\tau$. Thus it also governs the oscillatory behavior of the flame dynamics. Note, τ approaches infinity as the critical spacing is approached, and becomes a good measure for determining which propagation mode will dominate flame spread for a certain parameter space.

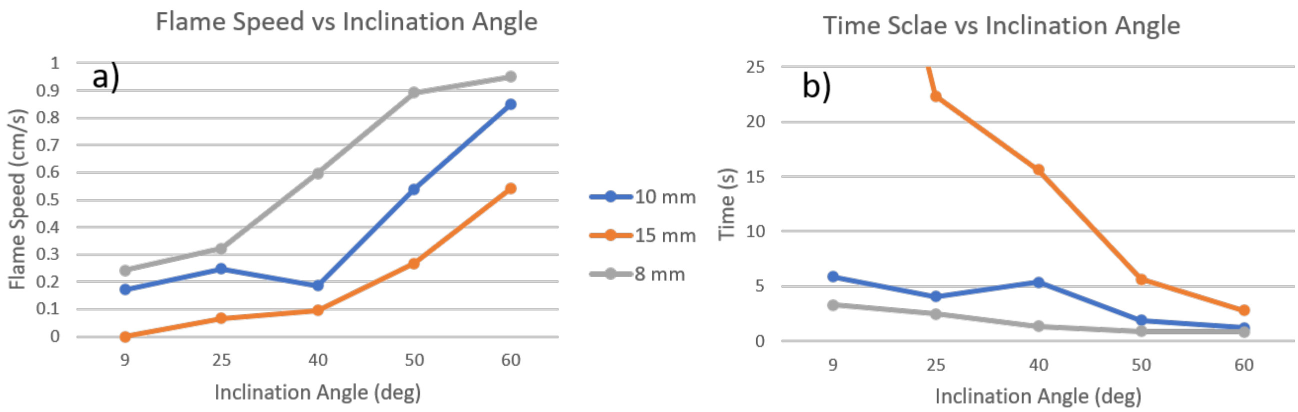


Fig. 6. Speed and time scale dependence on inclination angle, or slope, of the foam platform. a) flame speed for various spacing's as a function of angle. b) Characteristic propagation times for various spacing's as a function of inclination angle.

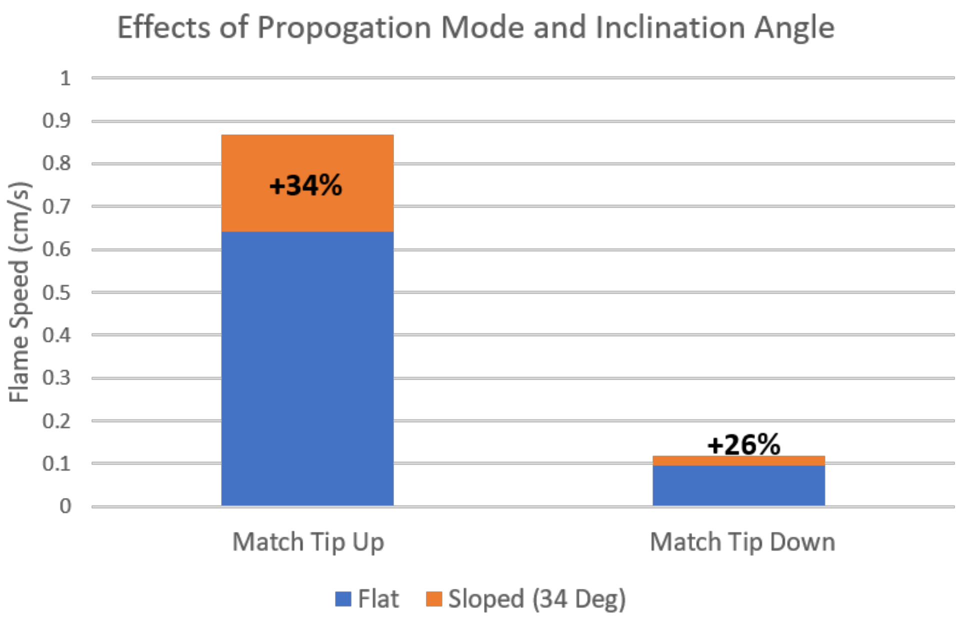


Fig. 7. Effects of match tips on flame speed. The left column shows the data for runs with match heads up. The blue is the speed in the flat case, the orange the speed in the 34 degree incline case, which was 34 % faster. The right column shows the data for runs with the match tips stuck in the foam. The blue is the flat case and the orange the 34 degree incline case, which was 26% faster.

There is also a nonlinear relationship between flame speed and inclination angle, regardless of match spacing. This can be readily seen in Figure 6. One can see another bifurcation in Figure 6a when looking at the 15 mm data series. This match spacing is too large for flame spread when flat, and is even too large at an angle of 9 degrees. However, a critical angle exists somewhere between 9 and 25 degrees which brings about a bifurcation point allowing for flame spread. Something like this will exist for all match spacing above the flat critical distance of 10 mm, and increasing the inclination angle in some ways can be thought of as reducing match spacing. It is also of interest to note the large angle case in Figure 6b. Regardless of match spacing, τ seems to converge as the steepness increases. This could be interpreted to mean that regardless of spacing, the convective mode dominates at some large angle limit.

Finally, the effect of the match tips themselves on flame speed was investigated. This was done by turning the matches upside down and sticking their coated tips into the foam. The bare sticks were then lit and the flame tracked and compared directly to the same parameter combination but with the match tips up. This configuration can be seen in the bottom image of Figure 3. The combination was a spacing of 5 mm for a flat and a 34 degree inclination case. The result of this study is shown in Figure 7.

The flame behaviour is completely different when the match heads are down. For starters, the resulting flame speed with match heads down is almost an order of magnitude smaller than that of heads up. Further, while there is still an increase in flame speed when inclined, the rate of increase due to the incline is on the whole smaller (34% vs 26%). These huge differences suggest an altogether different mechanism for flame propagation than those already discussed. This is also supported by the qualitative observations of flame spread with match heads down, as can be seen in the bottom of Figure 3. There is no longer wild, oscillatory behavior as the flame propagates, as in the convective dominated regime. Yet there is also not a step-wise propagation as seen in the radiative dominated regime. Instead the flame seems to calmly diffuse along the array of matches. Due to lack of overall flame movement this is likely also governed primarily by radiation, but is still dissimilar to the mode previously discussed.

Indeed this diffusive nature of flame spread has been documented in the literature. It has been argued that because flame front dynamics tend to self organize, or behave in a self similar and fractal like manner, they can be described similar to the spread of other diffusive systems - which similarly generate self-similar behavior [4,1,5]. Thus it could be argued that wild fires could be modeled using "dynamic percolation theory," as it is one of the most successful models of fractal like growth in diffusive systems. The observations of this work support that theory. Furthermore, the identification of a characteristic timescale for the diffusive mode of propagation could allow for the definition of an F_C, a "Fire Diffusivity Coefficient." In a 2D array F_C would be defined as the square of the match spacing divided by τ.

4 Conclusion

This work has several key takeaways. First, and perhaps most obvious is the non-linear dependence of flame speed on both match spacing and inclination angle. Within each of these non-linear relationships, it has been shown that bifurcation points exist, beyond which flame spread is or is not possible. More subtly, there has also been the identification of two main fire propagation modes, and characteristic times that correlate with each. A convective mode primarily caused by the match tip heads, and a radiation mode which dominates near the critical points. When using 2D match stick experiments to model wild fires, these heat transfer mechanisms and how they relate to actual wildfire phenomena must be clarified. Perhaps in some tinder-dry forests, trees may truly ignite such as a match tip, but it should be identified which types of forest fire lack this explosive convective mode, which in the match's case exists even in the absence of wind.

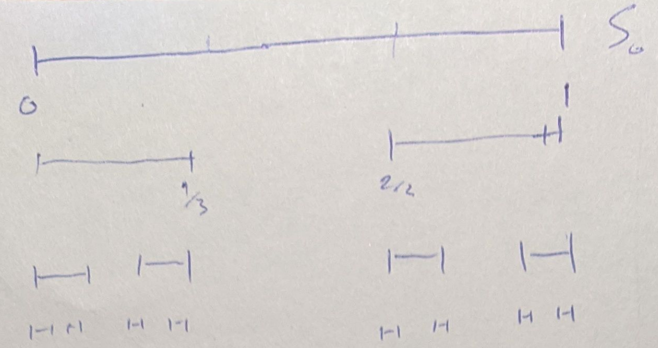
Further, it was identified that eliminating the effect of the match tips introduces a third mode of flame spread, more closely related to diffusive spreading. This mode has flame speeds and characteristic times of near order of magnitude differences than that of the other modes. How this mode reconciles, if at all, with real forest fires must be addressed in future desktop studies. Although these preliminary results further support the theory from literature that fires spread is a diffusive like system, this should be tested in future work using a Dynamic Percolation Model.

Future work should also attempt to decipher more the competition between radiation and convection in wildfires, and create analogous situations in desktop experiments. Further,the effect of the direction of the burn should be investigated. While wildfires typically start from ground fires (i.e. trees burn from the bottom up), desktop experiments do the reverse. Discerning how burning bottom up versus top down effects flame speed should be investigated before using desktop experiments to study wildfires.

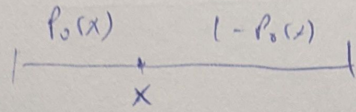
References

1. Caldarelli, G., Frondoni, R., Gabrielli, A., and et. al, *Percolation in real wildfires*, Europhysics Letters **56**, (2001) 510.
2. Alexander, Martin E., and Miguel G. Cruz, *Crown fire dynamics in conifer forests*,Synthesis of knowledge of extreme fire behavior **1**, (2011) 107-142.
3. Hoffman, C. M., Canfield, J., Linn, R. R., and et al., *Evaluating crown fire rate of spread predictions from physics-based models*, Fire Technology **52**, (2016) 221-237.
4. Malamud, B. D., Morein, G., and Turcotte, D. L., *Forest fires: an example of self-organized critical behavior*, Science **281**, (1998) 1840-1842.
5. Punckt, Christian, et al., *Wildfires in the lab: simple experiment and models for the exploration of excitable dynamics*, Journal of Chemical Education **92**, (2015) 1330-1337.
6. Morvan, D., *Physical phenomena and length scales governing the behaviour of wildfires: a case for physical modelling*, Fire Technology **47**, (2011) 437-460.

11.2.6



a) $P_0(x) = ?$



$0 < x < 1$

→ So if $x = 0.23$
LEFT OF x → 0 – .23
RIGHT OF x → .23 – 1

$P(x < 0.23) = 0.23 = x$
$P(x > 0.23) = (1 - .23) = .77$ → $P_0(x) = x$

b) S_1

$P_1(x) = ?$

SAY $x = 0.23$

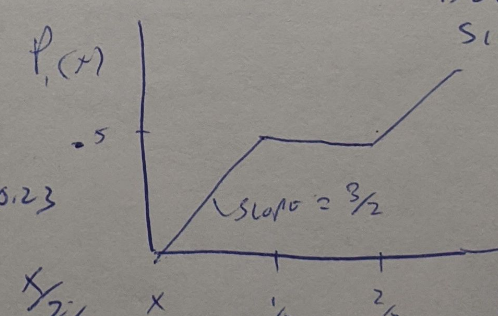
$x = 0.23$ — TOTAL SET IS NOT [illegible]

THEN TO THE RIGHT OF x

KNOW @ $x = 1/3 \to 2/3$ $P_1(x) = 0.5$ 50/50 CHANCE OF BEING ON EITHER SIDE

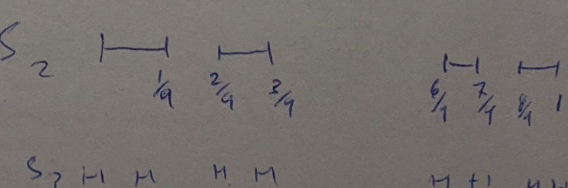
AMOUNT TO LEFT OF x = x
$x = 0.23$

AMOUNT TO RIGHT OF x =
$(\frac{1}{3} - 0.23) + \frac{1}{3} \to \frac{2}{3} - 0.23$

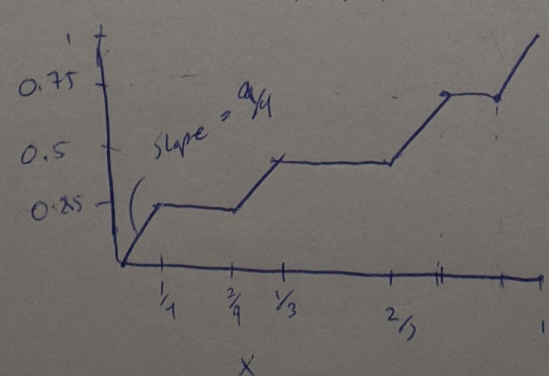


$P_1(x) = \frac{\text{LEFT}}{\text{TOTAL}} = \frac{0.23}{(\frac{2}{3} - 0.23) + 0.23} = \frac{x}{2/3}$

c) DRAW $P_n(x)$ FOR $n = 2, 3, 4$

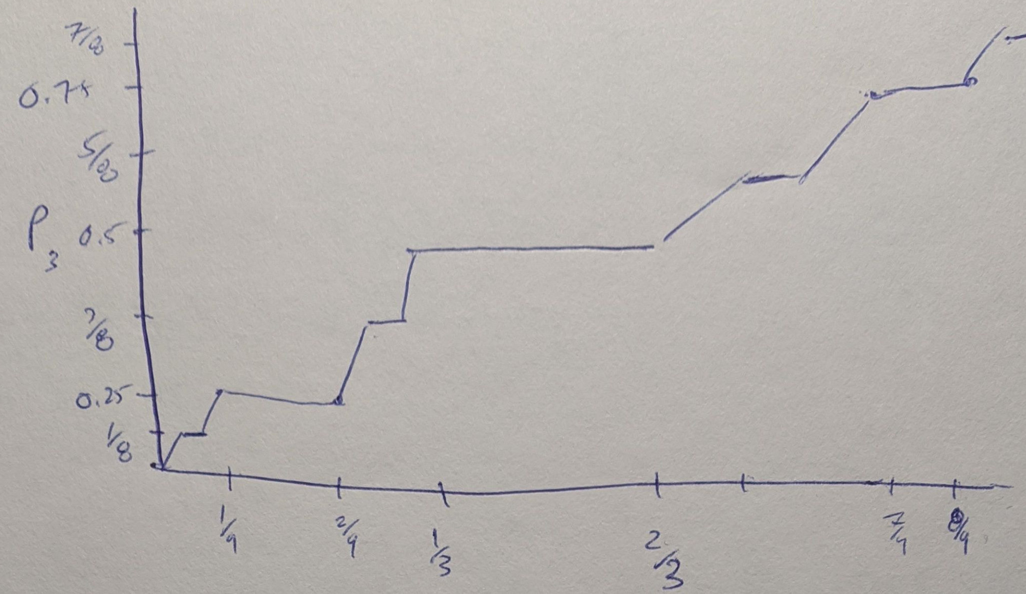


$P_2(x)$



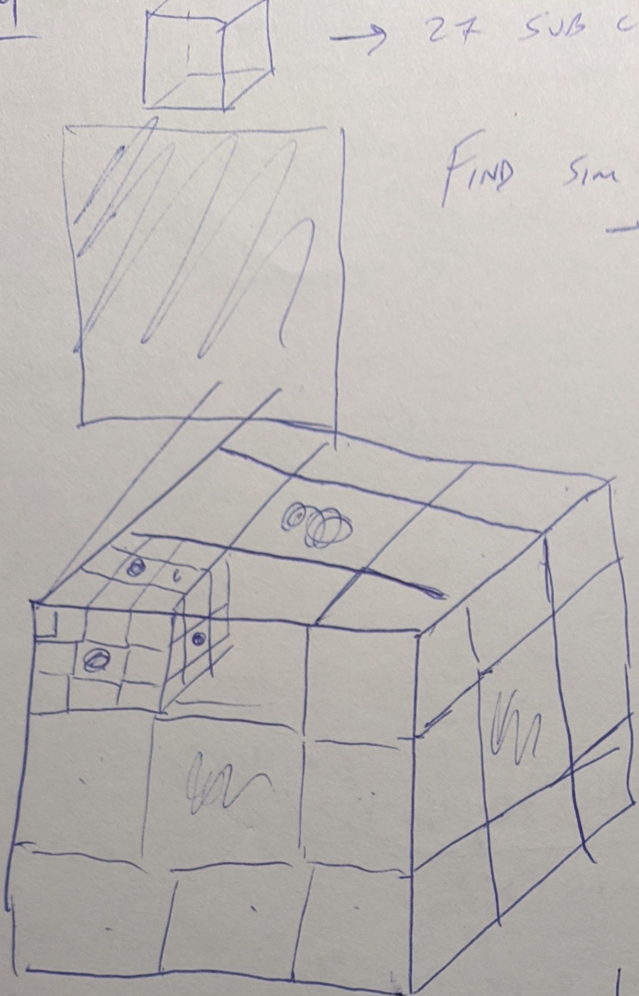
$P_2(x) = x / 4/9$

$n = 3 \rightarrow$ TOTAL $= 8/27$ SO $P_3(x) = \frac{27}{8}x$



C P_∞ THE OCCURRENCE OF STEPS BECOMES INFINITE, W/ INFINITELY SLOPED LINES CONNECTING THEM, THIS WOULD NOT BE DIFFERENTIABLE

11.3.9



→ 27 SUB CUBES

FIND SIM DIMENSION

$\rightarrow d = \frac{\ln m}{\ln r}$

m = # OF COPIES

SOLID CUBE → 1/3 SIZED CUBES

SO $r = 3$

EACH ITERATION @ EACH CUBE IS MADE INTO 27 CUBES

$27 - 6 - 1 = 20$

↑ EACH FACE ↑ CENTER CUBE

EACH ITERATION = 20 COPIES

$m = 20$

$d = \frac{\ln 20}{\ln 3} = 2.73$

c) SHOW 0 VOLUME

IF LENGTH OF 1ST CUBE IS L REMOVE 7 OF THE 27 SMALLER CUBES

$V_0 = V \rightarrow V_1 = \frac{20}{27} V \rightarrow V_2 = \frac{20}{27}\left(\frac{20}{27} V\right)$

⇓

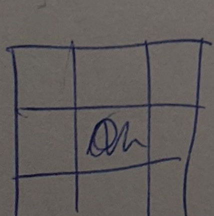
$V_n = \left(\frac{20}{27}\right)^n V$

AS $n \to \infty$ $V_n \to 0$

d) HYPERSPONGE

FOR 2D, $r = 3$

$m = 8$



FOR 3D $r = 3$

$m = 20$

$\frac{20}{8} = 2.5 \rightarrow 2.5 = \frac{m_4}{20} \rightarrow m_4 = 50$

$d_4 = \frac{\ln(50)}{\ln(3)} = 3.56$?

11.4.3 FIND BOX DIMENSION OF MENGER SPONGE

$S_1 = 20$ CUBES ~~LEAVES THE~~ 6 FACE

↳ 8 CUBES HAVE 3 EXPOSED FACES

12 CUBES HAVE 4 EXPOSED FACES

→ $8 \cdot 3 + 12 \cdot 4 = 72$ BOXES

W/ SIDE $\varepsilon = 1/3$

$S_2 \to 20^2$ CUBES

↳ 8^2 W/ 3 FACES

↳ 12^2 W/ 4 FACES

$N(\varepsilon) = 3(8^n) + 4(12^n)$

~~$d = \lim_{\varepsilon \to 0} \frac{\ln(3 \cdot 8^n + 4 \cdot 12^n)}{\ln(3^n)} = \frac{\ln(3(8)^n) + \ln(4(12^n))}{\ln(3^n)}$ & $\varepsilon = (1/3)^n$~~

$S_1 = 20$ cubes

EACH CUBE IS $1/3$ SIZE

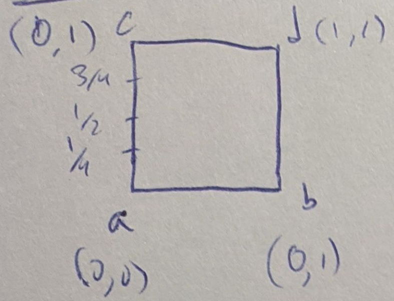
$S_2 = (20)^2$ cubes

↳ size $= (1/3)^n \to \varepsilon = 1/3$

$N = 20^n$

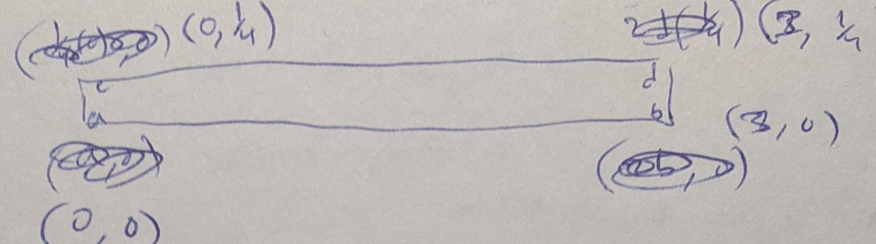
$$d = \lim_{\varepsilon \to 0} \frac{\ln(20^n)}{\ln(3)^n} \to \frac{n \ln(20)}{n \ln(3)} = 2.73$$

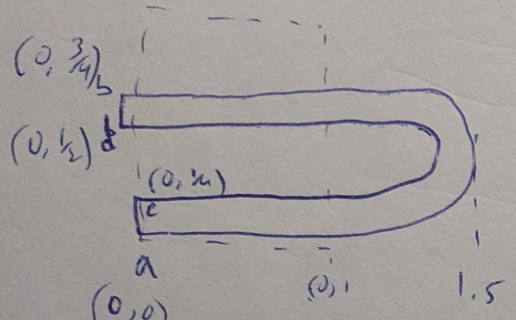
12.1.7

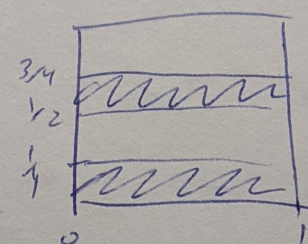


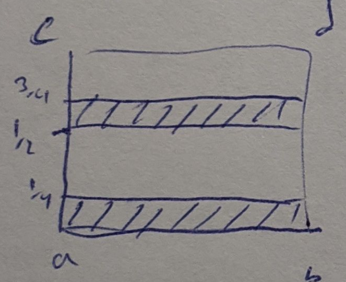
STRETCH BY FACTOR 3

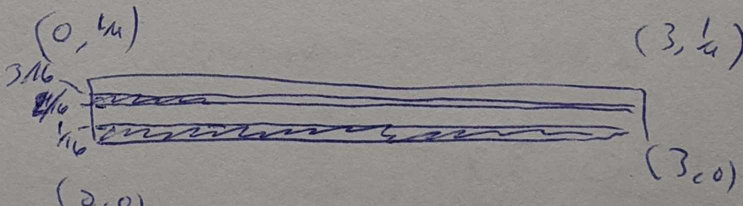
FACTOR BY 4

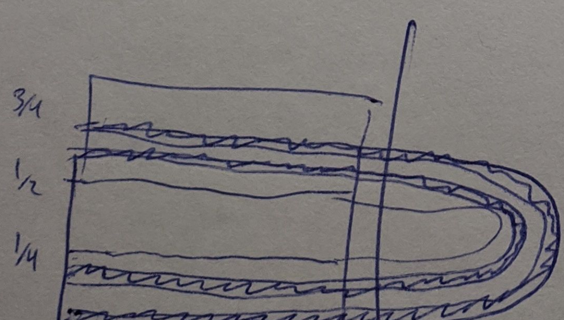


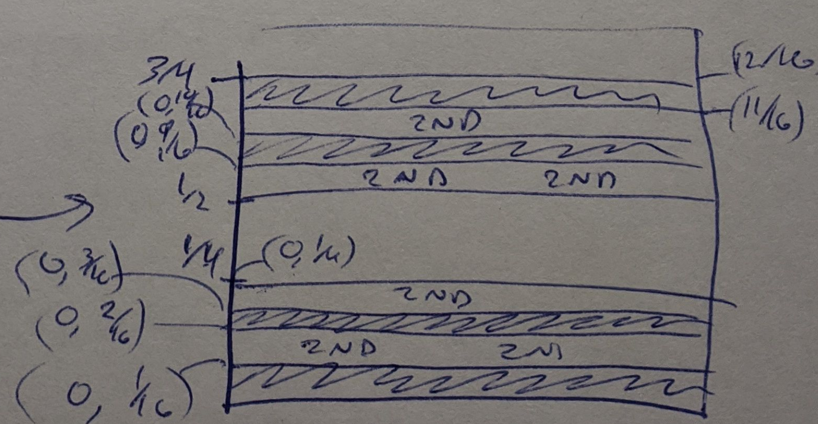












12.3.1 $b = 2$ $a\ (0 \to 0.4)$

$c = 4$

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

a) FIND HOPF BIFURCATION

SEE ATTACHED

FROM INSPECTION

HOPF BIF → $a \approx 0.33$

1ST PERIOD DOUBL → $a \approx 0.37$

b) $a \to 0.335$

SEE ATTACHED

c) $a \to 0.375$

SEE ATTACHED

12.4.3

SEE ATTACHED

A = 1.5 GIVES BEST RESULTS B/C
TRAJECTORY IS ~~SO~~ MOST CIRCULAR/
COVERS MOST SPACE

CAG

1.3

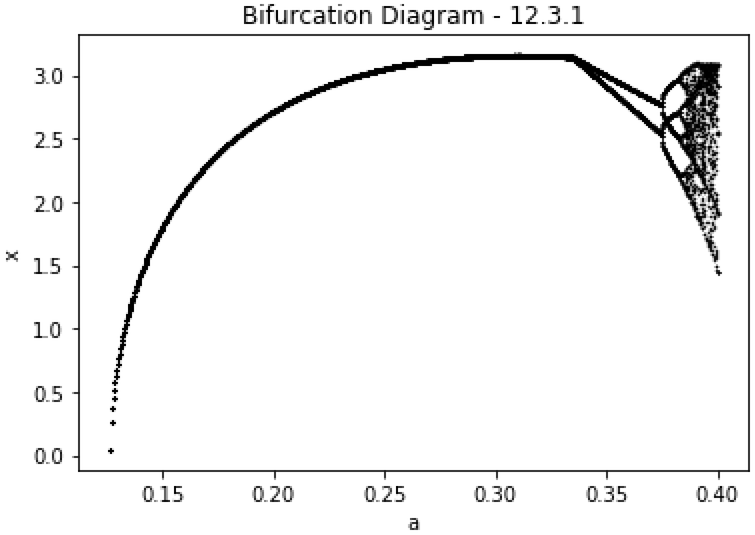
4000 POINTS IN A UNIT SQUARE
ALL W/ COORD (x_i, y_i)

HOW DETERMINE IF THESE ARE A PATTERN, OR JUST RANDOM

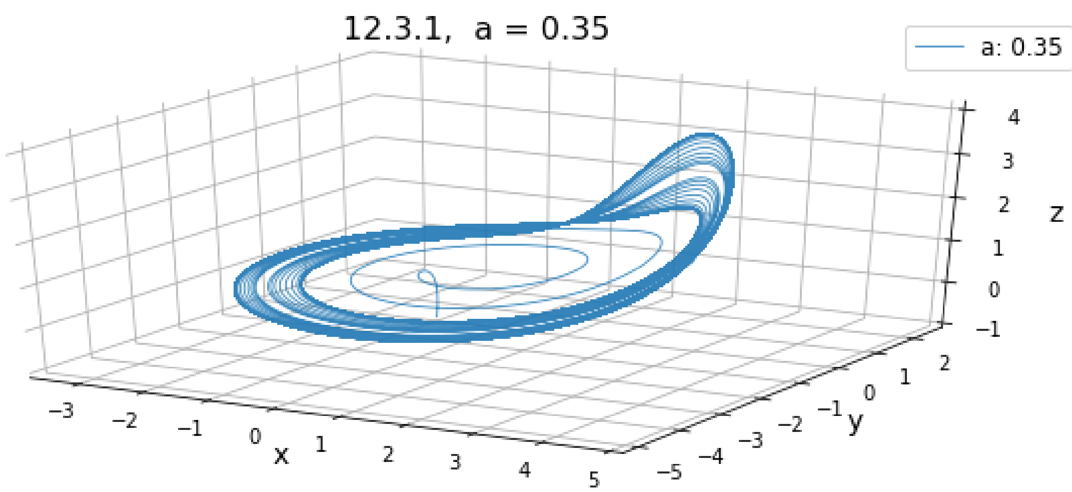
TO BE RANDOM THE PROBABILITY OF OCCURANCE OF A DATA POINT MUST BE THE SAME FOR EACH LOCATION. IF SOME LOCATIONS ARE MORE LIKELY TO HAVE POINTS, THIS LEADS TO STRUCTURE. THE DOTS MUST ALSO HAVE NO DEPENDENCE ON THEIR NEIGHBORS

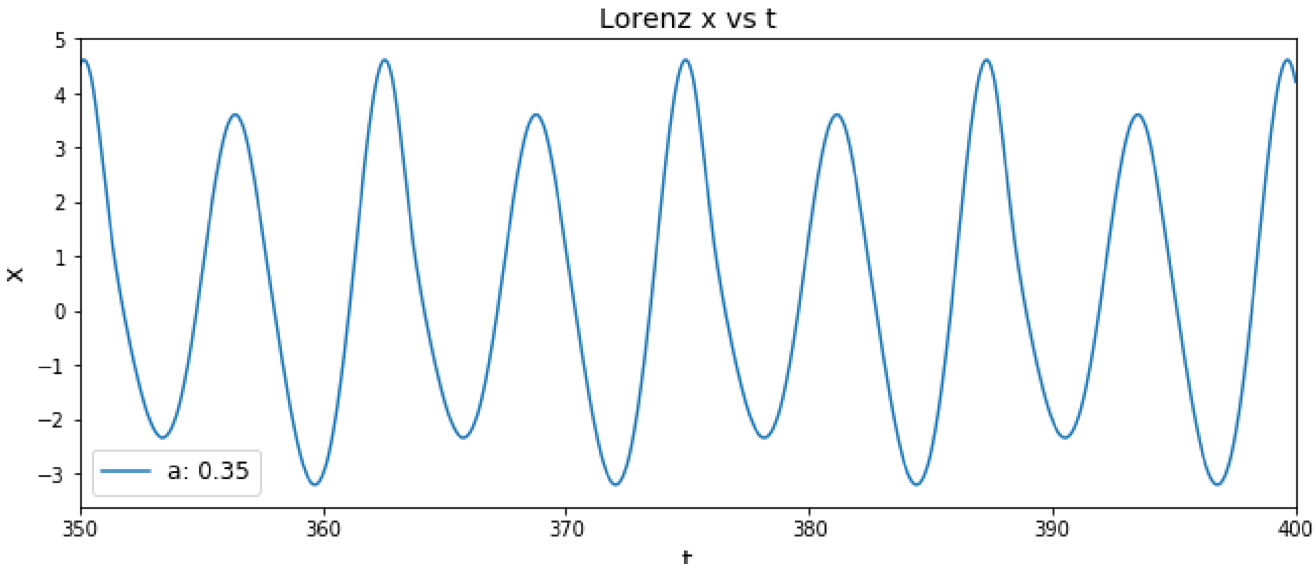
ONE COULD USE A CHI SQUARE TEST TO CHECK THE GOODNESS OF FIT OF THE SPATIAL DISTRIBUTION OF THE DATA TO A RANDOM DISTRIBUTION LIKE A POISSON DISTRIBUTION

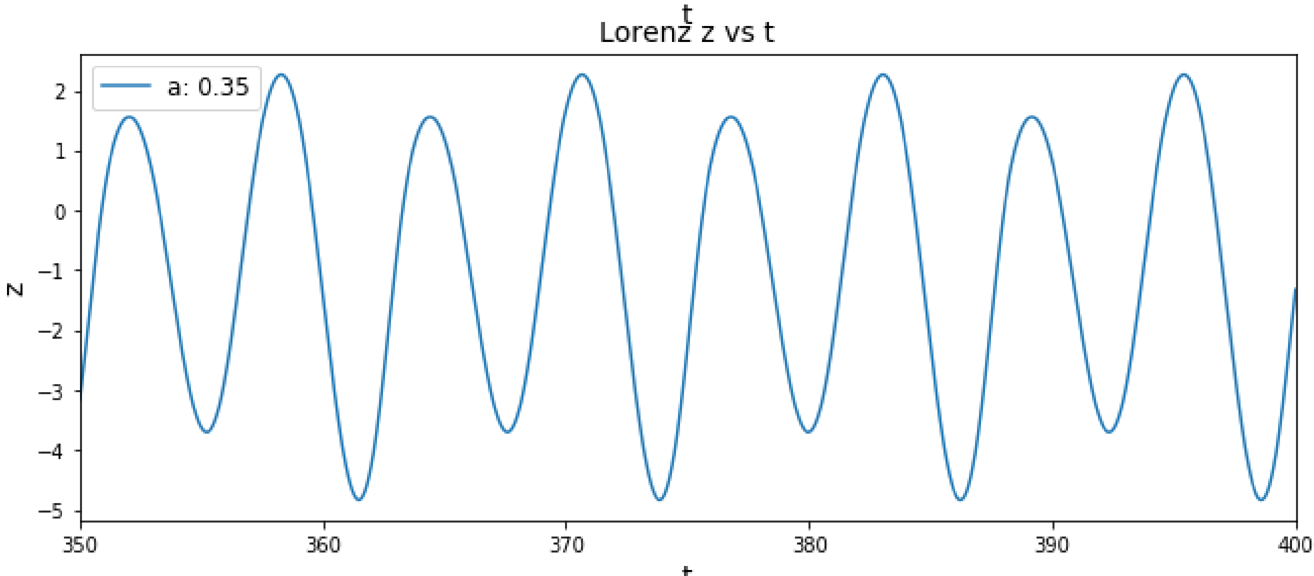
ANOTHER METHOD WOULD BE TO CROSS CORRELATE THE DATA POINTS & COMPARE THE CORRELATION COEFFICENTS W/ THAT OF A TRUE RANDOM DISTRIBUTION ON THE SAME SPACE

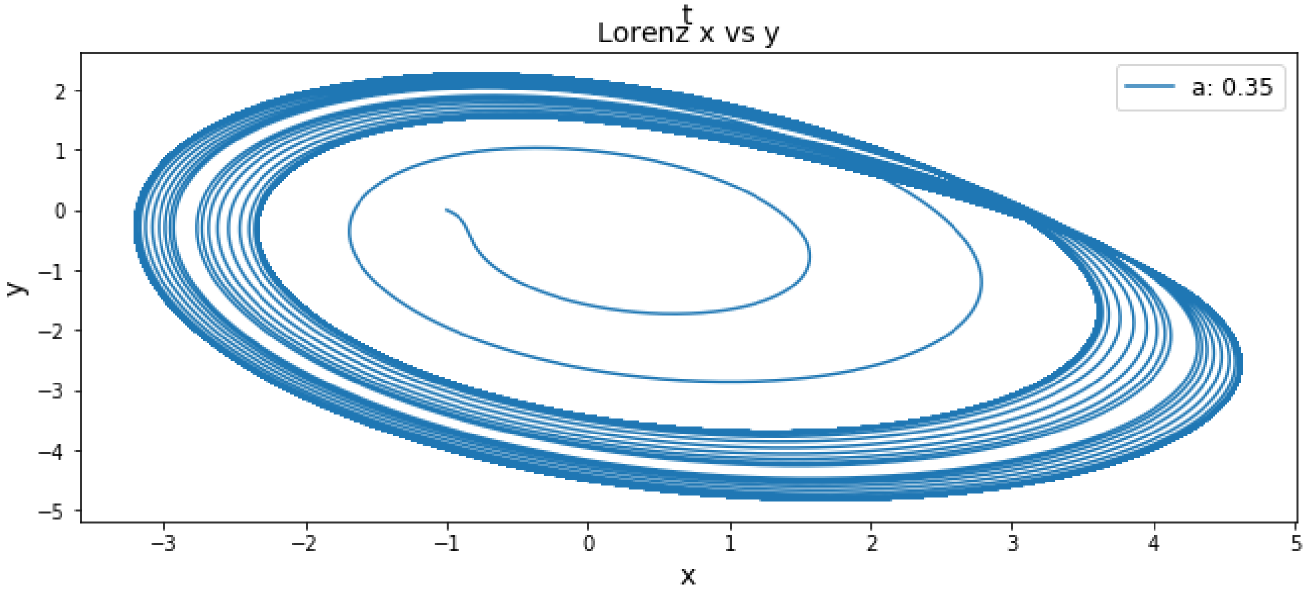


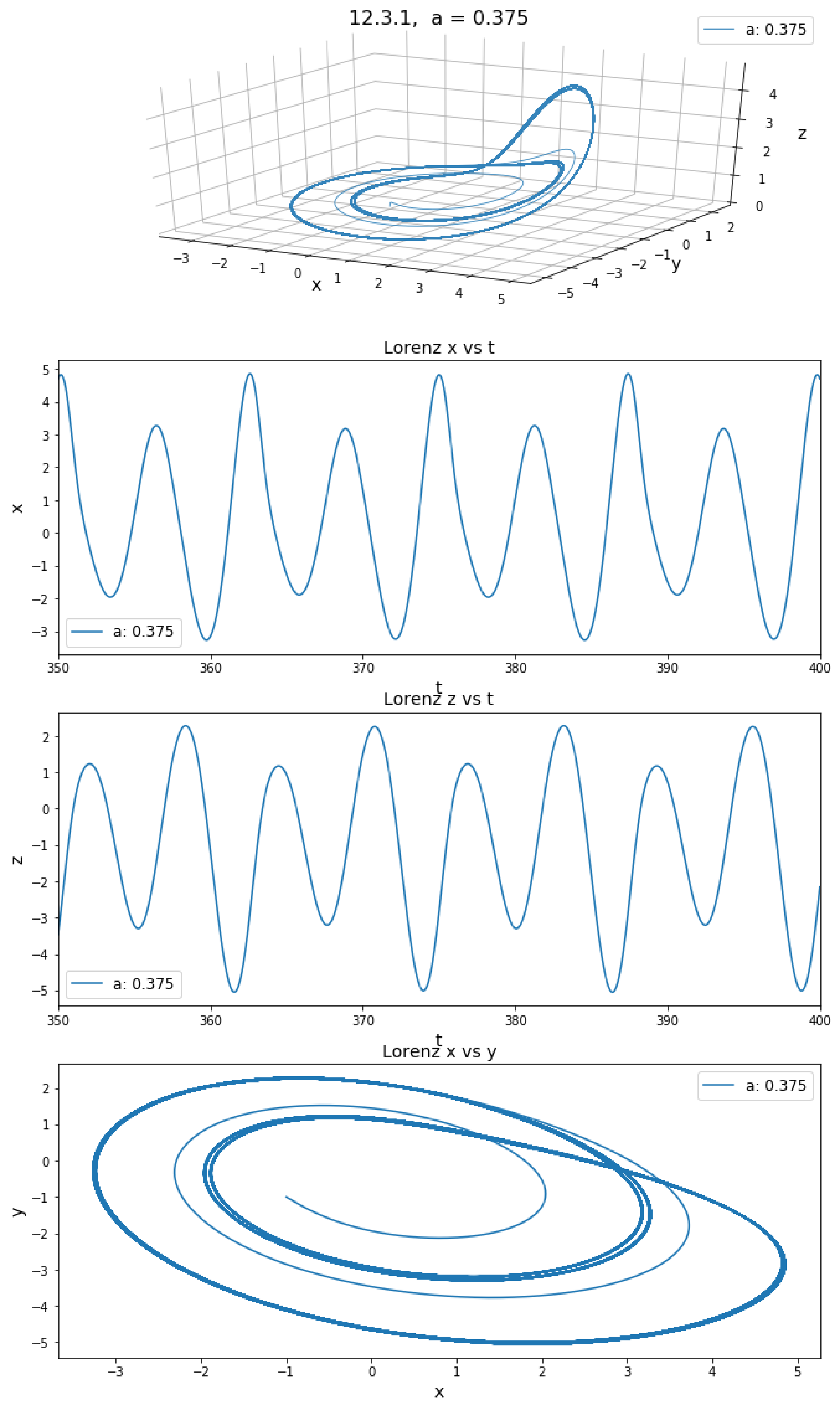
Bifurcation Diagram - 12.3.1
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.15
0.20
0.25
0.30
0.35
0.40
a


12.3.1, a = 0.35
a: 0.35
x
y
z

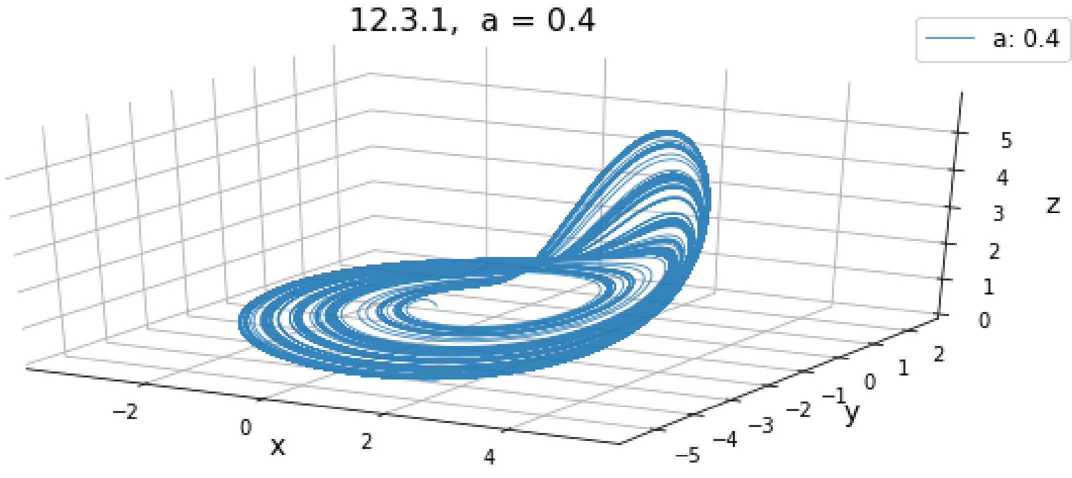

Lorenz x vs t
a: 0.35
x
t

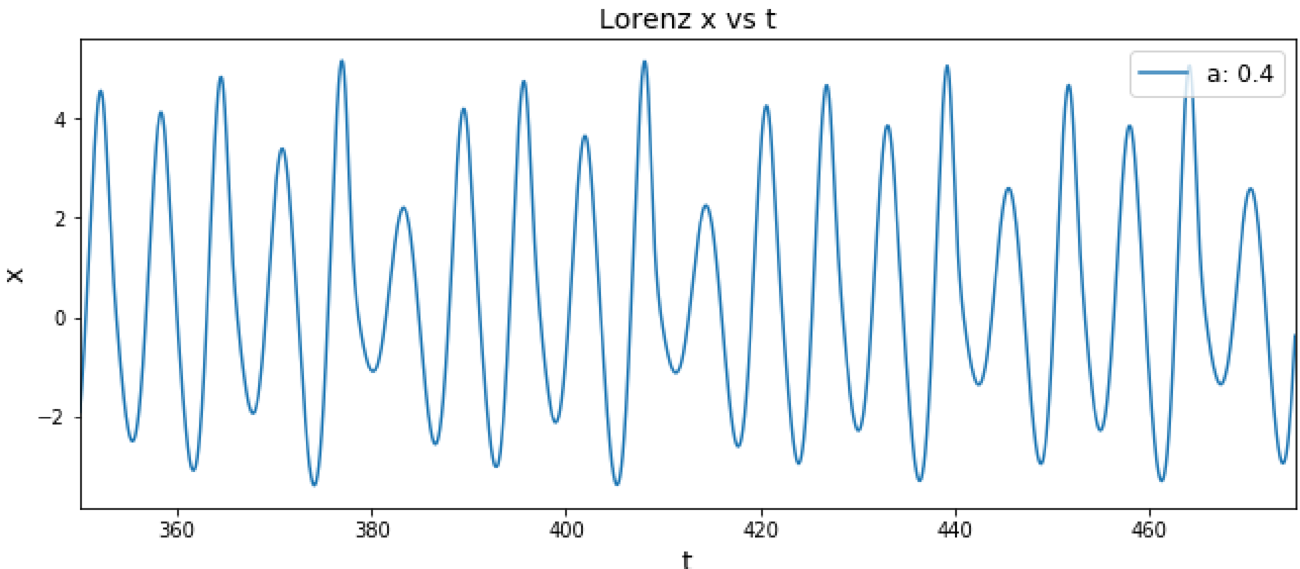

Lorenz z vs t
a: 0.35
z
t

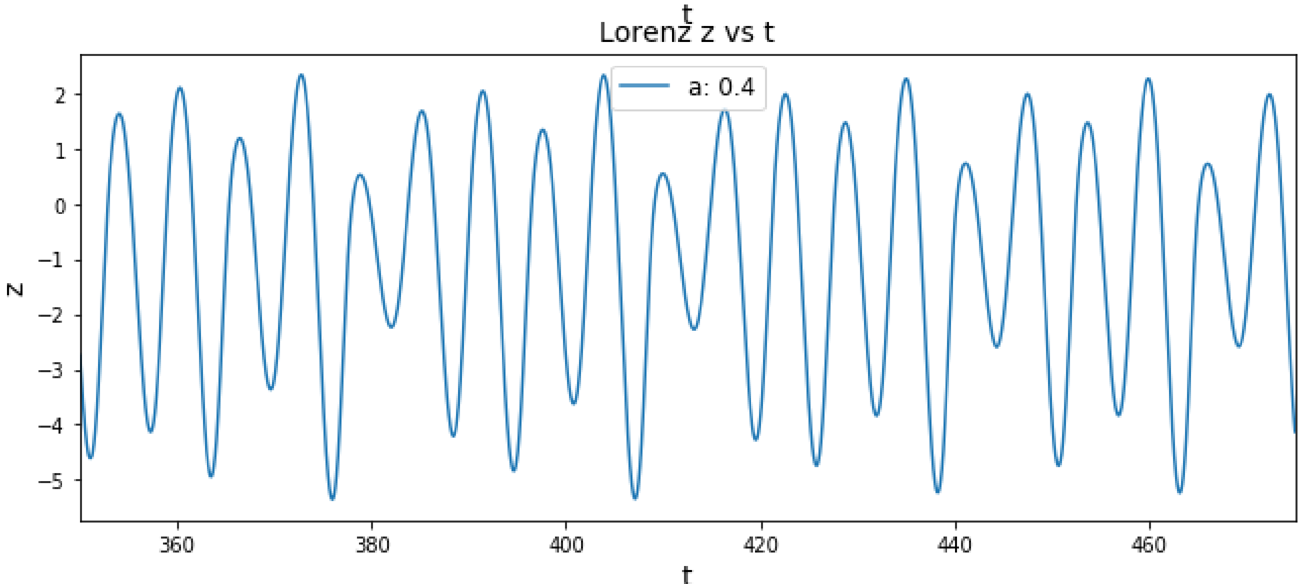

Lorenz x vs y
a: 0.35
y
x

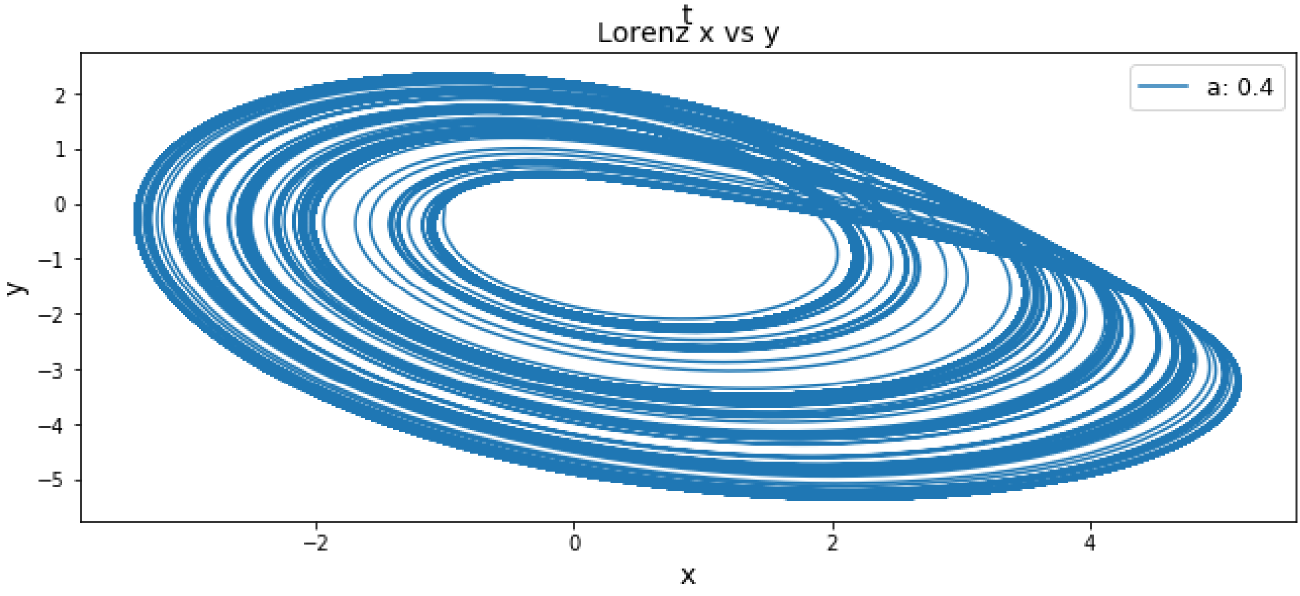


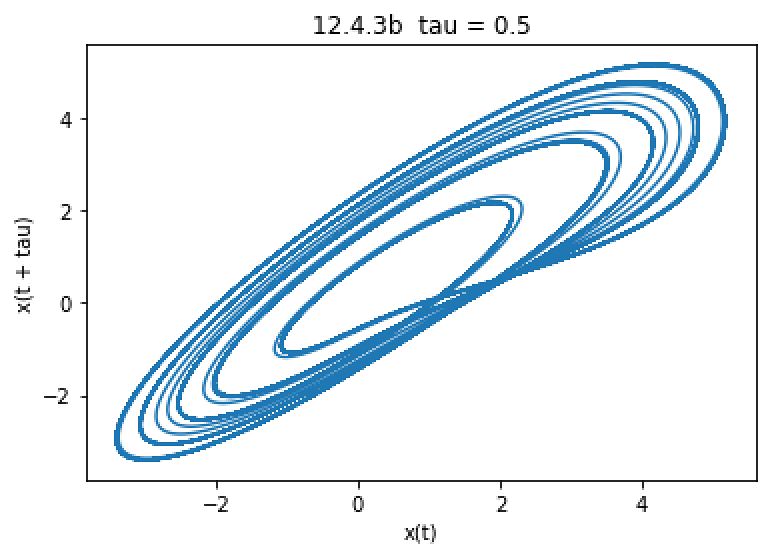
12.3.1, a = 0.375
a: 0.375
x
y
z
Lorenz x vs t
x
t
a: 0.375
Lorenz z vs t
z
t
a: 0.375
Lorenz x vs y
y
x
a: 0.375

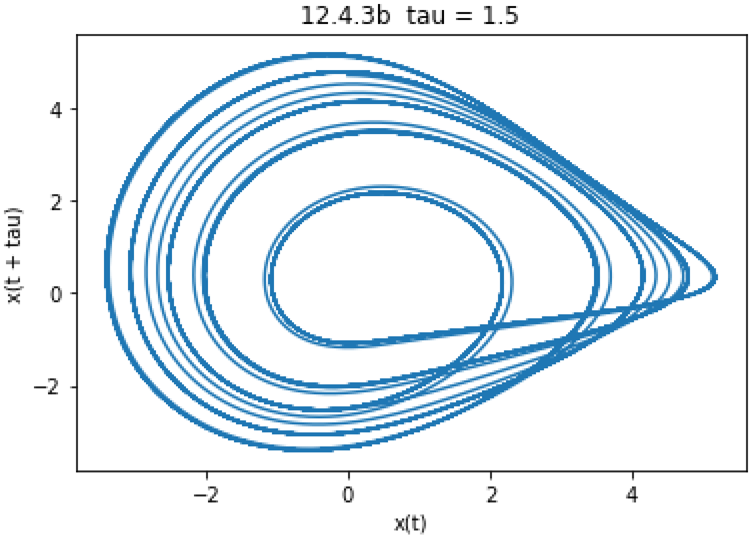

12.3.1, a = 0.4
a: 0.4
x
y
z

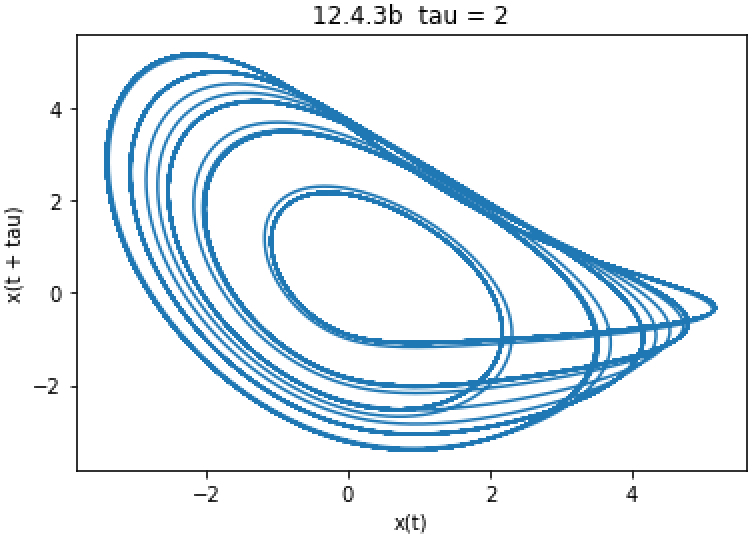

Lorenz x vs t
a: 0.4
x
t


Lorenz z vs t
a: 0.4
z
t


Lorenz x vs y
a: 0.4
y
x


12.4.3b tau = 0.5
x(t + tau)
x(t)
−2
0
2
4
−2
0
2
4


12.4.3b tau = 1.5
x(t + tau)
x(t)
4
2
0
−2
−2
0
2
4


12.4.3b tau = 2
x(t + tau)
x(t)
4
2
0
−2
−2
0
2
4

```
# -*- coding: utf-8 -*-
"""
Created on Mon Dec  7 11:02:13 2020

@author: 13302
"""




import      as
import                  as
import      as
from                import
import      as
from                          import
from              import

import
             1
#IC

    # [0,1,0],
    # [-10,10,0],

    # [random.randrange(-50,50),random.randrange(-50,50),random.randrange(-50,50)],

    # [20,0,-6]


# system parameters sigma, rho, and beta
                 0 0 375 1000
      2
         '12.3.1'
     4

# Integration time
             0
           900

                                                         500
       100

# lorenz system
# x, y, and z make up the system state
def rossler

    # positions of x, y, z in space at the current time point


    # define the 3 ordinary differential equations known as the lorenz equations




    # return a list of the equations that describe the system
    return
```

```
                          11 20
                     411            '3d'
                      412
                      413
                      414


# fig, (ax,ax2,ax3) = plt.subplots(3)
for   in
        # use odeint() to solve a system of ordinary differential equations
    # the arguments are:
    # 1, a function - computes the derivatives
    # 2, a vector of initial system conditions (aka x, y, z positions in space)
    # 3, a sequence of time points to solve for
    # returns an array of x, y, and z value arrays for each time point, with the initial values in


                             0


    # extract the individual arrays of x, y, and z values from the array of arrays
               0
               1
               2

    #detect when y switches sign


                                                           0


    for   in
        if             0


# plot the lorenz attractor in three-dimensional phase space

    # ax = fig.gca(projection='3d')
    # ax .gca(projection='3d')

                         1 1 1 1
              'y'          14
              'z'          14
              'x'          14
                         1 1 1 1
                         1 1 1 1
                       0 9            0 8         'a: ' str
                      ',  '  'a = ' str              16
                   12

              'Lorenz x vs t'          14
                                'a: ' str
               't'          14
             350 400
               'x'          14
                    12
```

```
            'Lorenz z vs t'          14
                              'a: ' str
           350 400
             't'          14
             'z'          14
                  12

            'Lorenz x vs y'          14
                    'a: ' str
             'x'          14
             'y'          14
                  12




for   in


    for   in
                                'k'          '.'   1
          'Bifurcation Diagram - 12.3.1'
           'a'
            'x'
```