Experimental Characterization of Nonlinear Dynamics from Chua's Circuit

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The nonlinear dynamics of Chua's circuit and its applications to communications is studied. Chaos is realized, and is investigated by calculating the Lyapunov exponent for the system. A three dimensional phase portrait for the characteristic double scroll is reconstructed from the two dimensional image, from which the correlation dimension is calculated. This experiment was done by designing Chua's circuit in the laboratory and comparing results to a simulation.

I. INTRODUCTION

There are many applications for chaotic systems, such as predicting nonlinear flow in the atmosphere, understanding the dynamics in cardiac arrhythmias, and handwritten character recognition. For many applications, it is often useful to generate chaotic behavior. This experiment focuses on generating a chaotic signal in a special circuit, called Chua's circuit, which can be useful in communications. Chaotic signals are usually broad-band, noise like, and difficult to predict[1]. This contrasts to the sinusoidal carriers often used. Their high power spectral density causes a high level of interference and enhances the probability of interception by other receivers[2]. A chaotic transmitter can send the desired information masked by chaotic signals. This masked signal is then transmitted to the destination, and the signal might be intercepted on the way, but would not be decoded without the masking chaotic signal. At the destination, another chaotic signal generator will be synchronized with the chaotic signal generator at the transmitter and thus produce the exact replica of the masking chaotic signal. The masked signal will then be demodulated by subtracting the masking signal from it to reveal the desired information hidden within[3].

Chua's circuit is the simplest autonomous circuit that can exhibit a bifurcation and generate chaos, because it satisfies the three criteria for displaying chaotic behavior: containing one or more nonlinear components, one or more locally active resistors, and three or more energy-storage devices. Initially, we wanted to achieve synchronization between two or more Chua's circuits acoustically[4].

However, after setting up our circuits and the acoustic component, we found out that the parts we had are too weak to communicate with the other circuit. Instead, in the process of building up the Chua's circuits and adjusting their parameters to find a chaotic signal, we found some interesting nonlinear behavior of the Chua's circuit. So we decided to focus on characterizing the nonlinear dynamics of the Chua's circuit.

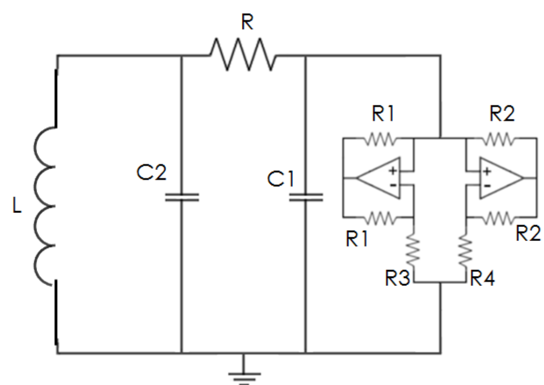

FIG. 1. A circuit diagram of Chua's circuit used for this experiment

II. EXPERIMENTAL SETUP

Chua's circuit can be built several different ways. This experiment replicated the circuit diagram in figure (1). It has four major components: (1) a variable resistor, whose resistance acts as the control variable, (2) two capacitors, whose electrical responses act as the dependent variables, (3) an inductor, and (4) Chua's diode, which is the source of the nonlinearity in the system. The electrical response of Chua's diode is shown in figure (2) and corresponds to the function g(V) in the differential equations (found under simulation).

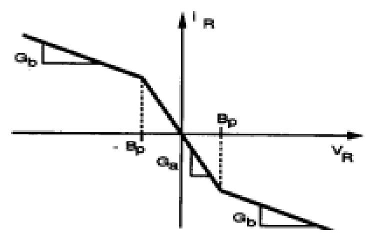

FIG. 2. The electrical response (IV curve) of Chua's diode - the nonlinear component. Function is piecewise-smooth

Figure (3) shows a picture of the circuit. Here there are actually two Chua circuits on one circuit board, but all experiments were carried out using only one. The project to study how two Chua circuits coupled together, either electrically or acoustically, was abandoned to study the dynamics of a single Chua circuit. As it turns out, the dynamics of a single Chua circuit are complex enough to merit further study.

Chua's circuit was hooked up to an oscilloscope and external computer that sampled the voltages across the two capacitors at 48,000 samples per second. Data was collected for several different values of the variable resistance.

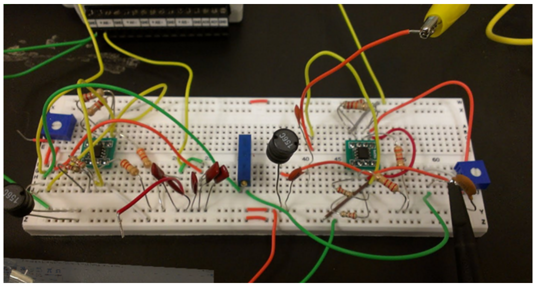

FIG. 3. Chua's circuit on a breadboard

III. EXPERIMENTAL RESULTS AND DISCUSSION

There are several different routes to chaos in Chua's circuit. In the experiment, the route observed was periodic behavior, followed by a screw attractor, and finally chaotic behavior, referred to as a "double scroll" in Chua's circuit[4]. This contrasts to period doubling, whose requirements of the function g(V) did not meet the specifications of our circuit[5]. The meaning of a screw attractor is most evident by looking at the phase space diagram for the two voltages in figure (4). Out of the periodic behavior, a single chaotic attractor was born as the resistance increased. This may be referred to as semi-periodic behavior, meaning that it is almost periodic, but has some chaotic aspects.

When the resistance was increased further, another attractor was born, forming a double scroll. The phase portrait for this behavior is shown in figure (5). Trajectories in phase space are confined to this double scroll and orbit around these two attractors in a complicated way. It is this complicated behavior that we wish to analyze.

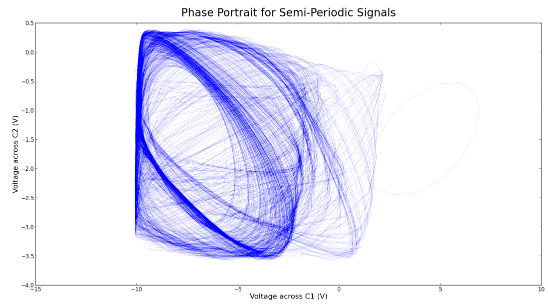

FIG. 4. Phase portrait observed between periodic behavior and double scroll.

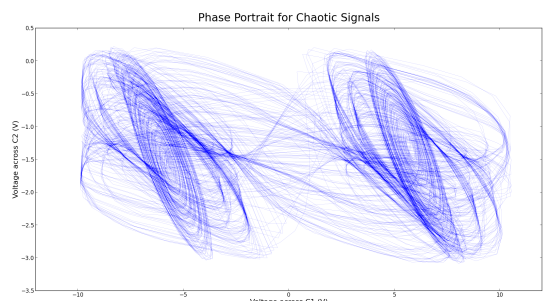

FIG. 5. Phase portrait observed after screw attractor. A new attractor is born

LYAPUNOV EXPONENT

The Lyapunov exponent is a measure of how two nearby trajectories diverge over time:

$$|\delta \mathbf{Z}(t)| \approx e^{\lambda t} |\delta \mathbf{Z}(0)|$$

If the value is positive, then any two nearby trajectories diverge exponentially, and chaos results. If the value is negative, then two trajectories converge exponentially and chaos is not observed. Knowing the Lyapunov exponent for a system is a good measure of how chaotic the system is.

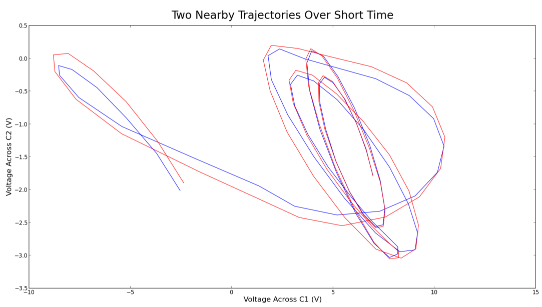

FIG. 6. Two nearby trajectories in phase space

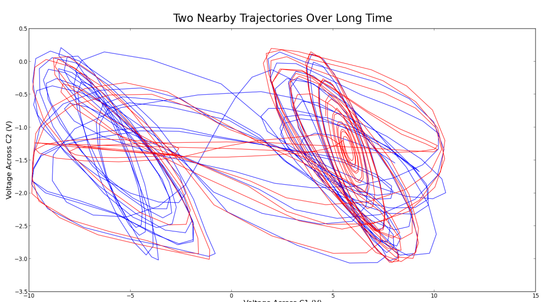

FIG. 7. The same trajectories from figure (6), but over a longer time.

Two nearby points were chosen in phase space by scanning the data and finding any two points that were separated by no more than 0.0075 V across each capacitor. These two points were of course separated in time by quite a bit. Finding points any closer would prove difficult or impossible despite having data collected over about ten seconds (an incredible amount of time for the dynamics of the circuit, which can change rapidly on the order of a millisecond). Two nearby trajectories in phase space are shown in figure (6) over a short time. In such a time, these trajectories are nearly locked on and only separate by a small amount in the end. Figure (7) shows the two trajectories over a long time. By this time, the trajectories have completely different behavior and no longer resemble each other. The system is chaotic, for a small separation in initial conditions has grown to something enormous. Figure (8) emphasizes the divergence by showing how the two trajectories compare on the voltage-time plot. They begin locked on, but after a few milliseconds they take on completely different behaviors.

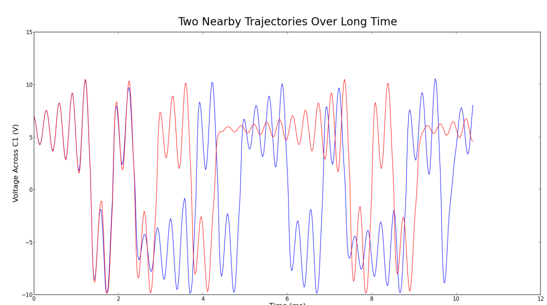

FIG. 8. Voltage-time plot for two nearby trajectories

A true description of two nearby trajectories requires n Lyapunov exponents in n-dimensional space. Each independent variable contributes to the total chaos of the system. This was not the method used here. Instead, the maximal Lyapunov exponent was measured. The maximal Lyapunov exponent is a measure of how the distance between two nearby trajectories in phase space diverges in time[6].

‘ Only a single value for the Lyapunov exponent is obtained using this method. However, the Lyapunov exponent depends on the initial conditions[6] and therefore depends on which attractor the trajectories start in, since the two attractors are not identical. So two Lyapunov exponents are expected, one for each attractor. In determining the Lyapunov exponent for each attractor, it was important to ensure that these trajectories did not leave the attractor they started in too quickly. Otherwise, the calculated value would be influenced by both attractors. Keeping in mind that the Lyapunov exponent varies in phase space, an average Lyapunov exponent can be defined for the whole system.

The Lyapunov exponent is determined by plotting the log of the distance between the two trajectories as a function of time. As a result, linear behavior is expected. One of these plots is shown in figure (9) and a line is fitted to the first part. There is a saturation point where the linear behavior disappears because the trajectories are confined to the attractor.

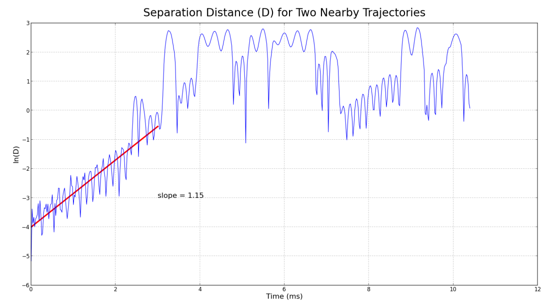

FIG. 9. Seperation distance for two nearby trajectories in phase space

This procedure was done for a number of initial conditions (each of which do not leave the attractor they start in too quickly) and an average was calculated for each attractor. The left attractor and right attractor have an average Lyapunov exponent of 1.0 and 1.7, respectively. These averages have a standard deviation of around 0.3, although the dependence of the Lyapunov exponent on position in phase space implies the distribution isn't normal anyways; there is some natural variance in the system. The variance obtained is due to this, but also due to the inexact measurement of the Lyapunov exponent, for there is no exact method for fitting a line to the curve.

In addition, a $t_{horizon}$ was defined from the average Lyapunov exponent of the system, 1.33.

$$t_{horizon} \sim O(\frac{1}{\lambda} ln \frac{a}{||\delta_0||})$$

Here, a is the tolerance level, chosen to be 0.1 V, and δ_0 is the uncertainty in measurement, chosen to be 0.001 V. The $t_{horizon}$ comes out to 3.46 ms. So after this time, the uncertainty has grown to 0.1 V. This is only a rough model.

ATTRACTOR RECONSTRUCTION

The idea behind attractor reconstruction is that the full 3-D phase space can be reconstructed from the measurement of only one variable. The method suggests that information about each independent variable is not self-contained[6]. The method relies on a delay time and reconstructs the data according to the following equations:

$$X = X(n) \quad Y = X(n+\tau) \quad Z = X(n+2\tau)$$

If the right delay time is chosen, then the attractor may be reconstructed. An optimal time choice is one that minimizes mutual information between the signal and its time-delayed signal[7]. The reconstructed attractor in 2-D is shown in figure (10). We see that a double scroll has been created in phase space, and only the voltage across one of the capacitors was needed. A double scroll is reconstructed in 3-D as shown in figure (11).

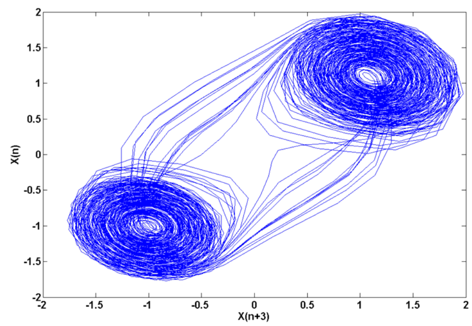

FIG. 10. The reconstructed attractor in 2D

The correlation dimension is a measure of the dimension of a set of points relative to the space it occupies. This is determined by measuring an average distance between each pair of points as the number of points tends to infinity. For the reconstructed attractor in 2-D, a correlation dimension of 1.8 was calculated.

IV. SIMULATION

In addition to the experimental component of this project, a simulation for Chua's circuit was also designed. The equations governing Chua's circuit are given by:

$$\frac{dx}{dt} = \alpha[y - x - g(V)]$$
$$\frac{dy}{dt} = x - y + z$$
$$\frac{dz}{dt} = -\beta y$$

Here, x is the voltage across C1, y

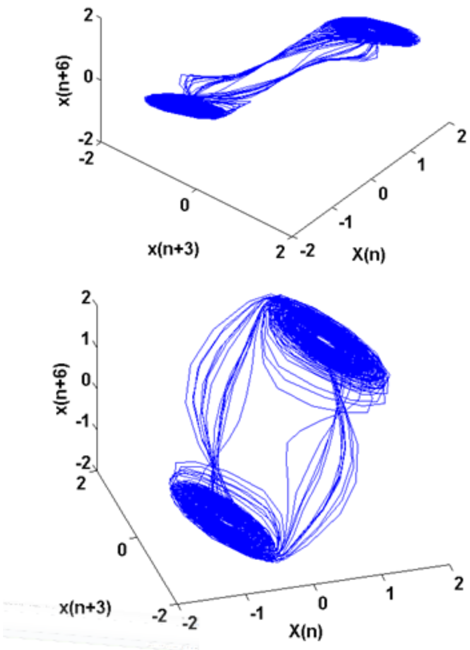

FIG. 11. The reconstructed attractor in 3D

is the voltage across C2, and z is the current in the inductor. These were solved using an adaptive Runge-Kutta fourth order method. The advantage of the simulation over experiment is that any of the three variables can be determined to a high degree of accuracy. This bypassed the need of an attractor reconstruction, but also enabled a comparison of simulated data to calculations. Although we did not do a rigorous comparison of the two, the simulation did generate results that were comparable to those obtained in the results.

Another advantage of the simulation is the ability to run the experiment for any value of the resistance and tweak other values of the circuit components. This enabled the simulation to construct a bifurcation diagram for a circuit that meets the criteria for period doubling, a behavior unobservable in our experiment. The diagram is shown in figure (12), and plots the possible voltage values as a function of the capacitance.

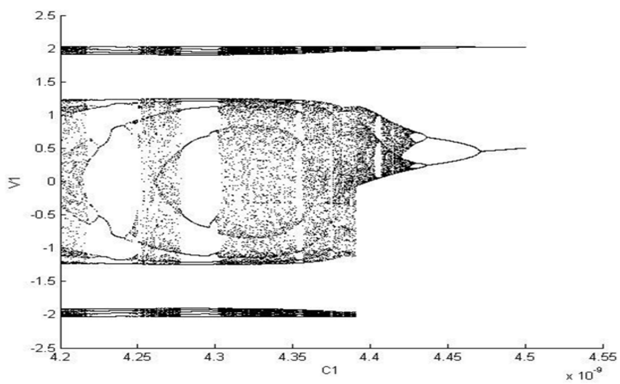

FIG. 12. Bifurcation diagram for Chua's circuit

V. CONCLUSION

Chua's circuit has within it many complex behaviors that demonstrate the aspects of chaotic behavior. Because it is so simple to build and play around with, the circuit is widely used to generate chaotic output in the laboratory. Although it is so widely used, it is still not fully understood, for the complexities of the circuit are far too deep. The purpose of this experiment was to demonstrate and apply various methods in nonlinear dynamics to the circuit in hopes of shedding some light on the complexities observed.

There are a host of other things that can be done with Chua's circuit. One thing we would like to do is to construct greater comparisons between simulation and experiment. For instance, it is known that the Lyapunov exponent varies depending on the trajectory chosen, i.e. it has a spatial dependence. The simulation could answer such questions as 'How large is this variance for different points all across the attractor?' This could then be compared to the variance obtained in calculating the Lyapunov exponent, and possibly explain why such a variance was observed.

REFERENCES

[1] K. Cuomo, A. Oppenheim, and S. Strogatz, "Synchronization of lorenz-based chaotic circuits with applications to communications," Circuits and systems II: Analog and digital signal processing, IEEE Transactions on **40**, 626–633 (1993).

[2] P. Arena, A. Buscarino, L. Fortuna, and M. Frasca, "Separation and synchronization of piecewise linear chaotic systems," Physical Review E **74**, 026212 (2006).

[3] H. Dedieu, M. Kennedy, and M. Hasler, "Chaos shift keying: modulation and demodulation of a chaotic carrier using self-synchronizing chua's circuits," Circuits and systems II: Analog and digital signal processing, IEEE Transactions on **40**, 634–642 (1993).

[4] Y. Harada, K. Masuda, and A. Ogawa, "Dynamical behavior of acoustically coupled chaos oscillators," Fractals-an Interdisciplinary Journal on the Complex Geometry **4**, 407–414 (1996).

[5] L. Chua and L. Huynh, "Bifurcation analysis of chua's circuit," in *Circuits and Systems, 1992., Proceedings of the 35th Midwest Symposium on* (IEEE, 1992) pp. 746–751.

[6] S. Strogatz, "Nonlinear dynamics and chaos: with applications to physics, biology, chemistry and engineering," (2001).

[7] A. Fraser and H. Swinney, "Independent coordinates for strange attractors from mutual information," Physical review A **33**, 1134 (1986).