

---

# Vortex Rings and Non Linear Dynamics

---

**Peter Lebedev**

Department of Computer Science  
Georgia Institute Of Technology  
Atlanta, GA 30332  
plebedev3@gatech.edu

## 1 Introduction

### 1.1 Background

Vortex rings have been known for an extraordinarily long time in human history. From smokers blowing rings to smoke signals created by fire, these dynamics have been in our human history for quite some time. In this paper we outline a baseline of initial results and an argument for new surprising results.

### 1.2 Motivation

Vortex rings are found in many important systems of note. For example, turbulence and turbulent flow are a commonly studied problem in fluid dynamics. Vortex rings can appear in these turbulent systems and their formation are of an important note as to why. Modeling the fluid dynamics with rain and associated turbulent systems could be important as well, as Vortex rings appear right underneath collision of raindrops with the surface of water. The human heart form a vortex ring while pumping blood, and understanding their formation would be important towards creating replacement organs for those in need. Vortex rings also appear underneath helicopters, which can be associated with dangerous conditions due to a loss of lift. Kids and excited physicists play with vortex ring guns, where a diaphragm and a cylindrical barrel with a hole create an expulsion of air to create these doughnut shaped rings. In addition, vortexes can form when coughing through a mask as well under specific conditions.

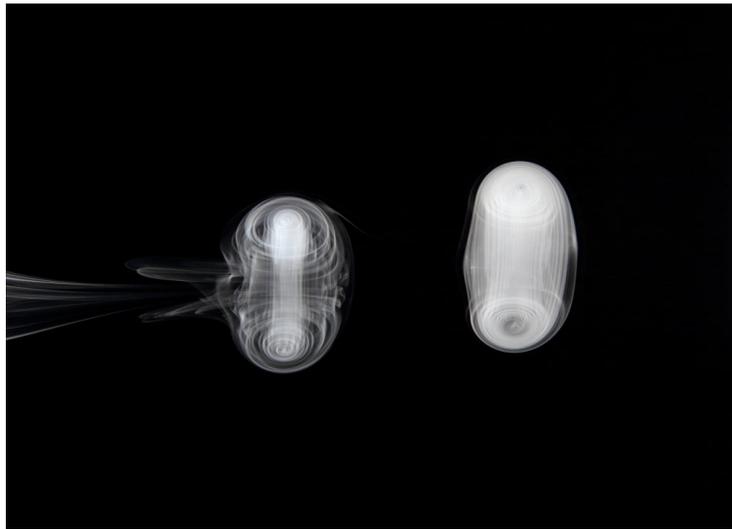


Figure 1: Vortex Ring

## 2 Vortex Ring Dynamics Review

Given the easily made vortex cannon, we can start with the formation of vortex rings through a simple model of this system. We can use a simple slug model, as provided by Wilson, 2003. This system has a much faster flow of fluid created within the slug, into a stationary fluid outside the slug. Viscous friction from the flow and particles colliding into the fluid (or a barrier) cause an initial slowdown of outer layer expelled by the slug. As the fluid moves, it creates a wake where the parts that were slowed collect behind and re enter the ring. This creates a ring like flow denoted by K.

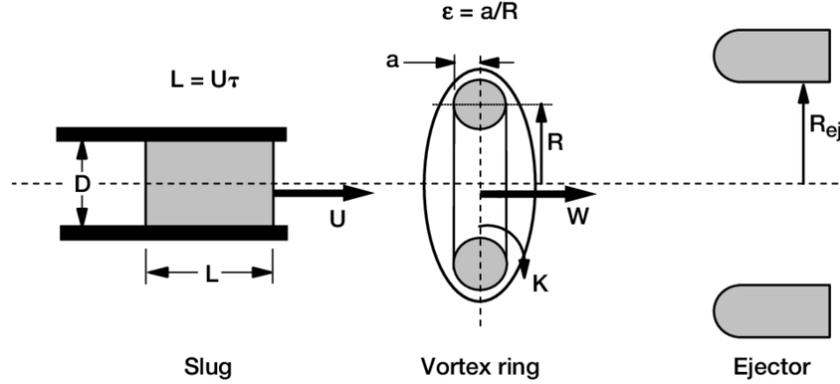


Figure 2: Vortex Ring Slug Model. Provided by Wilson 2003: A Simple Model of Pulsed Ejector Thrust Augmentation

### 2.1 Background of fluid dynamics and Analysis By Nitsche 2006

Nitsche gives a good analysis of voracity of fluids, a quantity measured by the local fluid rotation about an axis. Given a velocity field  $u(x, t)$ , the voracity is given by  $\omega = \Delta \times u$ . This velocity is expanded to give the following:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0) + D(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}_0) \times (\mathbf{x} - \mathbf{x}_0) + O(|\mathbf{x} - \mathbf{x}_0|^2)$$

where

$$D(\mathbf{x}_0) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \nabla\mathbf{u} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}.$$

Where the first term gives the translational component, the second gives a stretch matrix, of which the eigenvalues are of note. These eigenvalues can give whether the fluid will stretch or compress. This can show whether the fluid will form sheets or tubes. The last is given by the rotation.

Nitsche also gives a time evolution of voracity given by the Navier-Stokes equations. This is:

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\omega} \cdot \nabla\mathbf{u} + \nu\Delta\boldsymbol{\omega}$$

Given,  $\nu$  the viscosity, and the gradient. This can be used as the basis for our analysis as well.

She also mentions that these small rings are unstable. If it is perturbed, it could form hexagonal like rings. This is known as azimuthal instability. This is also usually put in terms of the Reynolds number.

### 3 Materials and Methods

#### 3.1 Approach

We created a simple slug model, made out of 2 PVC pipes. One was 3 inches in inner diameter, and longer than the 4 inch pipe. The larger one was 4 inches in inner diameter. The inner smaller pipe was sealed, while we fixed rings of varying diameters on the outer pipe. The inner pipe was placed inside the outer pipe to create the slug, and bungee cords were attached from the end of the inner pipe to the outer pipe. The outer pipe was attached securely to a desk with duct tape, to prevent recoil effects.

By pulling back on the inner pipe, and letting go, we create an impulse of air through the pipe, and out through the aperture into the room. This impulse may or may not form rings.



Figure 3: Our vortex slug model. The top two images are two snippets of the side camera video of pulling back of the inner pipe at certain displacements. The bottom is the resulting expulsion of air

We varied 4 parameters - the tenseness of the bungee to control the velocity of the piston, the stopping point of the inner piston ( $l_0$ ), the initial displacement of the inner piston, and different diameters of the outer piston. Different bungee cords, with varying  $k$  constants were used, as well changing the attachment at different points of the outer piston, to vary the velocity. Screws were placed at different locations inside the outer piston to vary the stopping points. Initial displacements were varied through markings on the inner piston. Different pvc caps with holes drilled of varying diameters were attached with duct tape onto the outside of the piston.

Each was explored through the upper half of the parameter space. We took a set of bands, a diameter of the ring, and a  $l_0$ . We varied the displacement until we did not get rings anymore, and a bit further so we can explain where rings form.

Videos were taken of the inner pipe when launched to get the velocity. Top down video was also taken to get any dynamics, and side video was also taken. For the side video, marked displacements were put on a wooden board and captured on video to get translational velocities from the video data.

#### 3.2 Reasoning

We created this slug model, as it was relatively simple to make, and could also be scientifically understood through the works of other people It is well documented in the scientific literature. We

initially wanted to simulate COVID spread in fluids, so we can simulate cough spread with it as well. We had enough parameters to vary and to create new results as to why rings form. Given the online semester, creating a new apparatus with water would be much more difficult and expensive. We also weren't sure if we would meet up, so we created this so we can quickly create a experiment that we can run at home. This is an inexpensive, quick option for results.

#### 4 Qualitative Results

displacement	D = 1 1/8	tenseband	D=2 (tense)	D = 2 (new band)	D = 2 (tense)	D = 1 1/8 (tense)	D = 3/4 in (ter D1 *1/2)	D = 3/4	D = 2	
0.5	ring	ring	x	ring	x	x	ring	ring	ring	x
1	ring	ring	x	ring	x	ring	no ring	ring	no ring	x
1.5	ring	ring	x	ring	x	ring	x	ring	x	x
2	ring	ring	ring	ring	ring	ring	x	ring	x	ring
2.5	no ring	no ring (ghost)	x	x	x	no ring	no ring	ring	no ring	x
3	no ring	no ring	x	ring	x	no ring, splitting patt	split ring	ring	split ring	x
3.5	no ring	x	x	x	x	x	ring	ring	ring	x
4	no ring	no ring	ring	ring	ring	x	no ring	ring (messy)	no ring	ring
5	x	x	x	x	x	x	no ring	split two	no ring	x
5.5	x	x	x	x	x	x	x	split two	x	x
6	no ring	no ring	ring	ring	ring	x	no ring	no ring	no ring	ring
6.5	x	x	x	ring	x	x	no ring	x	no ring	x
7	x	x	x	ring	ring	x	x	no ring	x	ring
7.25	x	x	x	x	ring	x	x	x	x	ring
7.5	x	x	DAMAGED	ring	x	x	x	no ring	x	ring
8	x	x	x	no ring	x	x	x	x	x	no ring
8.25	x	x	x	no ring	x	x	x	x	x	x
8.5	x	x	x	no ring	x	x	x	x	x	x
9	x	x	x	no ring	x	x	x	no ring	x	x
10	x	x	x	no ring	x	x	x	x	x	x
12	x	x	x	no ring	x	x	x	x	x	x
$l = 1/2$ in	$l = 1/2$ in	$l = 1/2$ in	$l = 1/2$	$l = 1$ in	$l = 1$ in	$l = 1$ in	$l = 1$ inch	$l = 1$ inch	$l = 3$ in	$l = 3$ in

This table shows which parameters we explored which form and do not form rings. Green highlighted text forms rings, while red highlighted text did not form rings.

The different stopping points were highlighted at the bottom, while the different diameters are mentioned at the top. The change of bands was mentioned at the top.

From this, we can infer several parameters that form and do not form rings. Columns 1 and 2 show the same diameter, with the band placed at different points as to vary the velocity of the piston. Regardless of how fast the piston is moving, we still did not form rings at at the same displacement.

Varying  $l_0$ , our piston stopping point, also did not effect ring formation. Comparing the fourth and last columns, we see rings form at the same displacements.

The interesting point is the ring at the seventh column, or  $D=3/4$ ,  $l_0 = 1$  The ring forms, then doesn't form and then forms again as the displacement is increased. This will be more quantitatively discussed in the next section.

We've also observed azimuthal instability towards the end of the ring's lifecycle, before the ring broke apart. This is probably because the vorticity is decreased as the ring slows down due to friction with the air around it and so is more vulnerable to perturbations from the environment. The small perturbations create large scale ripples in the ring, however capturing it on camera and doing analysis with videos is difficult due to exposure of faint rings.

Azimuthal instability is still being researched, and is quite difficult to measure and explain. From More and Saffmans work, we know that there is a strain from the the distant parts of the ring, and the vorticity of the core of the ring. We've qualitatively noticed that the ring does expand the further it goes, and so the strain is increased as the ring expands, causing Azimuthal instability.

## 5 Quantitative Results

### 5.1 Graphs of Displacement over Diameter Versus Circulation Ratios

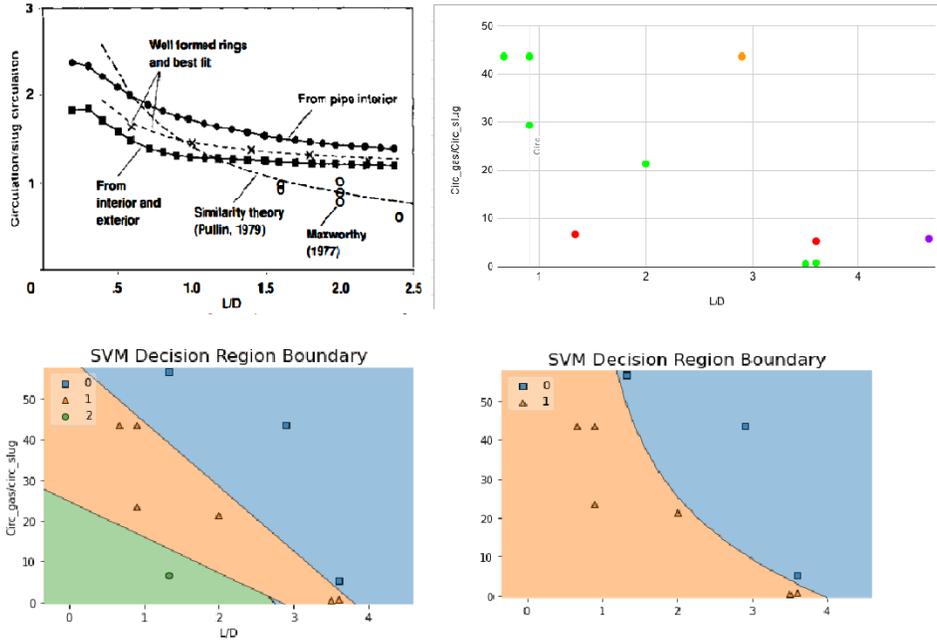


Figure 4: Top left: Previous work done by Shariff and A Leonard, a graph of 2 different models and previous results. Top Right: Our data for a consistent (yet slightly erroneous) set of points. Green denotes points that form rings, red and orange did not, and purple is the ring that forms again at the 7th column. Bottom left: a linear separating lines of fit between our points, ignoring the purple point. Orange region denotes our ring forming area. Green and Blue denotes our non forming area. Bottom right, a polynomial fit comparing the ring forming points and non ring forming points for circulation values directly above.

### 5.2 Previous work

Previous work highlighted by Shariff and A Leonard gives us approximations of circulation using vorticity, the key term of the formation of vortex rings. Circulation is the line integral of a vector field around a closed curve, while vorticity is a field that describes the local behavior of near some point. They also bring into account the important quantity  $L/D$ , or our displacement divided by the diameter of the ring. These circulation ratios and  $L/D$  values are dimensionless.

Shariff and A Leonard combine results and analysis from three sources. From the figure in the top left, we have a dashed line of fit, which corresponds to the equation:

$$\frac{\Gamma}{\Gamma_{slug}} = 1.14 + 0.32(L/D)^{-1}, \quad L/D > 0.6.$$

Which corresponds to a equation of fit towards vortex rings created in liquids, by Didden 1979

The other equation by the dashed line:

$$\frac{\Gamma_{gas}}{\Gamma_{slug}} = 1.41(L/D)^{-2/3}$$

Represents the work of Pullin 1979, who used similarity theory, and is a better fit for higher Reynolds numbers

To approximate circulation values, we use Gharib, Rambod, Shariff's approximation, of vorticity, assuming that radial flow is zero.

$$\frac{\Gamma_{slug}}{dt} = \int_0^{t_{piston\ stop}} \omega u_x d\sigma \approx \int_0^{t_{piston\ stop}} \frac{\partial u}{\partial \sigma} u_x d\sigma \approx \frac{1}{2} U_p^2(t)$$

By integrating the vertical velocity squared of both the piston and the gas for the same time, we can get a decent approximation of the circulation of the gas, and fit our points.

### 5.3 Quantitative Methods for a set of results

To get the velocities of the gas and the piston, we used both the side video and the piston/cannon firing video. We took each frame, marked its displacement from the initial displacement. We then calculated velocity from multiplying this from the frames per second of each video, then squared this quantity and divided it by 2. We then took a box integral of the quantity of each frame, and summed the result. We did this for a set of results that was consistent enough in parameters used with each other, such as not to scale the error too far.

We also tried optical flow to get the velocities directly. Sadly our FPS was not enough, and we could not repeat the experiment, as discussed later.

These results were then plotted on the scatter plot in figure 4, in the top right corner.

To get dividing lines between our circulation values, we employ a Support Vector Machine (SVM), which projects the data points into a higher dimensional space through a kernel trick. This is implemented in scikit-learn in python. In a SVM the margins between different classes of points are minimized subject to a regularization term  $c$ . Support vectors derived from points are taken along the decision boundary, to minimize the margin between the classes, We employed various high values of  $c$ , and used a polynomial kernel of dimension 3 for the bottom right figure, and a linear kernel for the bottom left figure.

### 5.4 Quantitative Analysis and Error

While our results do not fit the curve all too well, and vary by a factor 20, our general direction of the points do. The endpoint behavior for well formed rings by the pulling curve:

$$\lim_{L/D \rightarrow \infty} \frac{\Gamma_{gas}}{\Gamma_{slug}} = \lim_{L/D \rightarrow \infty} 1.41(L/D)^{-2/3} = 0$$

While ours does as well, as noted by the rings at  $L/D$  at 3.5 The polynomial SVM(top right) also starts curving upwards towards these two points, showing that our analysis somewhat starts fitting the Pulling curve.

We also have points above and below the curve, and as shown by the linear SVM,(bottom left) we have a bound of the region where well formed rings occur, which shows that a potential line of fit or a parameter space can be formed from vortex rings.

Our error and differences comes from the low FPS taken from these videos. The approximation of the integral becomes much more when only 1-2 frames of the piston are launched. While the side video's FPS (30) doubles the piston's FPS (18), both are still prone to this error. These FPS's can probably over approximate our circulation values of the gas at low displacement, as the piston will be more accurate, while at higher displacements and higher piston velocities, over approximate our piston circulation values, as it travels at much higher speeds while also starting at a larger displacement. We think that these errors is somewhat proportional, so while its hard to compare to other peoples results, it still should still have the same behavior as the Pullin curve.

We also took a video with better frames captured of the piston and gas at  $L/D = 3.75$ , and got a circulation ratio of 0.53, which roughly approximates the pullin curve at their circulation ratio of 0.58

### 5.5 High Reynolds numbers and the unexpected ring

We are also dealing with high Reynolds's numbers, enough for the turbulent flow regime. We calculated our Reynolds's numbers through the approximation given by Shariff and A Leonard,

$$R_e = \frac{\Gamma}{v}$$

where  $v$  is viscosity. To approximate viscosity, we assumed our gas was pure water vapor (which it mostly was), and got  $v = .00217$ .

Our highest Reynolds number for the gas was  $R_e = 4058$ , which was the purple outlier point, as mentioned previously in the qualitative section, which does not seem to fit any of the curves above!

We also got slightly better accuracy of the piston and gas from frames captured due to the slightly slower velocity the gas and piston was moving at, due to having the smallest diameter out of all our rings (3/4 inches), and the small displacements we needed to use.

Our Reynolds number, since most of our gas comes out of the piston cylindrically, is enough to enter the turbulent regime. Our velocities of this data point during ring formation also increased and then decreased dramatically, like it was hitting a brick wall. This leads us to the hypothesis that the ring formation comes from Kolmogorov's Theory of Inertial Turbulence, which explains how larger eddies break up and transfer their energy to smaller eddies. The dramatic slow down from hitting the gas, allows energy from the initial eddy formed, which doesn't form a ring, to break up into a smaller eddy, resulting in a smaller Reynolds's number, forming a ring.



Figure 5: 6 selected frames, highlighting the slowdown of the purple outlier point, then showing the larger eddy formation, and subsequently forming a slightly smaller ring

We think that the previous curves for well formed rings don't take this into account, and could lead to unexpected ring formation in turbulent, chaotic, fluids! More non linear dynamics should be explored to capture these results in more detail, as we could have another area in the parameter space that can form rings that are sensitive to initial conditions. (They did not form all the time!)

For a new problem in Strogatz's book, if theories, and subsequent approximations were developed to make why this point arises, and developed simple enough for an introduction to non linear dynamics, it would be a interesting problem to explore the parameter space, while also noticing that rings can form in another region that's not completely expected! The 3-d Navier stokes equations aren't completely solved, and the vorticity arising from them is probably very sensitive to initial conditions.

These new rings can explain why perhaps unexpected vortexes can form under helicopters. With the probably high Reynolds numbers, this could lead to better safety!

## 6 Discussion, Error and Future Work

Other than the low FPS of the video taken, I also tried to use optical flow to create a more accurate vorticity approximations (assuming the horizontal flow captured by the laser is even along the radial axis, and then actually doing the integral around the ring) however, the frames per second were not enough as denoted by the figure down below. I wanted to do more experiments, however, I had to quickly leave off campus before thanksgiving due to my family getting COVID. Due to the pace of the COVID semester, I would have loved to use a high FPS camera, but we would need to use three of them, which I know can get quite pricey, and putting the expensive camera near the near-ballistic

slug cannon we made might not turn out so well. In addition, we need high resolution and high FPS cameras for optical flow which not all cameras have, and would increase the price incredibly.

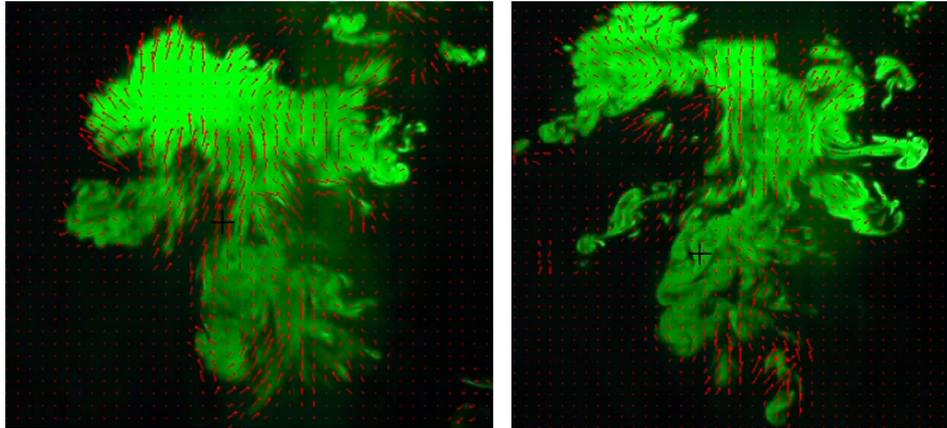


Figure 6: Optical flow, the arrows are not accurate enough to pick up on enough vorticity, and so we would need better FPS

In addition, this would take a LOT of code, which would be an entire research project. Clustering, and subsequent identification of the ring would need to be done, or a manual selection and tracking of the selection for each frame would need to be done, both which can take up hundreds of hours of work if we wanted to take this integral. However it can be done, and could lead to dramatically better fluid dynamic experiments as shown by the figure done. We could use this with vertical lasers to better observe azimuthal instability, for example, as its still being researched!

We also had tens of thousands of frames of video we could have parsed, but due to time constraints, and the excessive amount of time this took, we could do this more in the future.

We also could have explored the lower bound of our parameter space. If we reduced the velocity by a significant amount, we also would have seen low circulation values, and might have seen slightly different behavior at lower Reynolds numbers. Rings would form less, however, we would need a lot more bands, or a accurate piston to control the velocity.

In terms of research, I do want to see if we can create rings in spaces rings theoretically as of now shouldn't form. Kolmogorov's theory of inertial turbulence could be to explain. We should create a more controlled experiment with better cameras, and better bands to explore as to why this happens.

Other errors to take in account is that we needed to patch up our pipe with duct tape, as it did break. Our screws did also bend and snap all the time, which lead to slight variations in our stopping point, however we did observe that our stopping point did not change our values in which rings did not form, so this might be negligible. Our bands also did stretch out, changing our velocity, but since it was captured on camera, we still captured velocity. We also could have gotten the k constants, and then the velocities, but due to the low quality of the bands, and them stretching out, this could have led to error as well.

## 7 References

(1) (2) (3)

### References

- [1] F. Kaplanski, S. Sazhin, S. Begg, Y. Fukumoto, and M. Heikal, "Dynamics of vortex rings and spray-induced vortex ring-like structures," *European Journal of Mechanics - B/Fluids*, vol. 29, no. 3, pp. 208 – 216, 2010. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S0997754610000038>
- [2] M. Shusser and M. Gharib, "Energy and velocity of a forming vortex ring," *Physics of Fluids*, vol. 12, no. 3, pp. 618–621, 2000. [Online]. Available: <https://doi.org/10.1063/1.870268>
- [3] J. Wilson and R. DeLoof, "A simple model of pulsed ejector thrust augmentation," 09 2003.
- [4] Moffatt, H.K. (2015), "Fluid Dynamics", in Nicholas J. Higham; et al. (eds.), *The Princeton Companion to Applied Mathematics*, Princeton University Press, pp. 467–476
- [5]. <https://www.sciencedirect.com/science/article/pii/S1364032107000500>: :text=The
- [6] "The Feynman Lectures on Physics Vol. II Ch. 3: Vector Integral Calculus". [www.feynmanlectures.caltech.edu](http://www.feynmanlectures.caltech.edu). Retrieved 2020-11-02.
- [7] Gharib, Rambod, Shariff. A universal time scale for vortex ring formation. *Fluid Mechanics* 360, 1998.
- [8] F. Kaplanski, S. Sazhin, S. Begg, Y. Fukumoto, and M. Heikal, "Dynamics of vortex rings and spray-induced vortex ring-like structures," *European Journal of Mechanics - B/Fluids*, vol. 29, no. 3, pp. 208 – 216, 2010. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S0997754610000038>
- [9] M. Shusser and M. Gharib, "Energy and velocity of a forming vortex ring," *Physics of Fluids* vol. 12, no. 3, pp. 618–621, 2000. [Online]. Available: <https://doi.org/10.1063/1.87026>
- [10] J. Wilson and R. DeLoof, "A simple model of pulsed ejector thrust augmentation," 09 2003.
- [11] K Shariff, and, and A Leonard "Vortex Rings" *Annual Review of Fluid Mechanics* 1992 24:1, 235-279
- [12] Moore Derek William and Saffman Philip Geoffrey 1975, The instability of a straight vortex filament in a strain field *Proc. R. Soc. Lond.* A346413–425