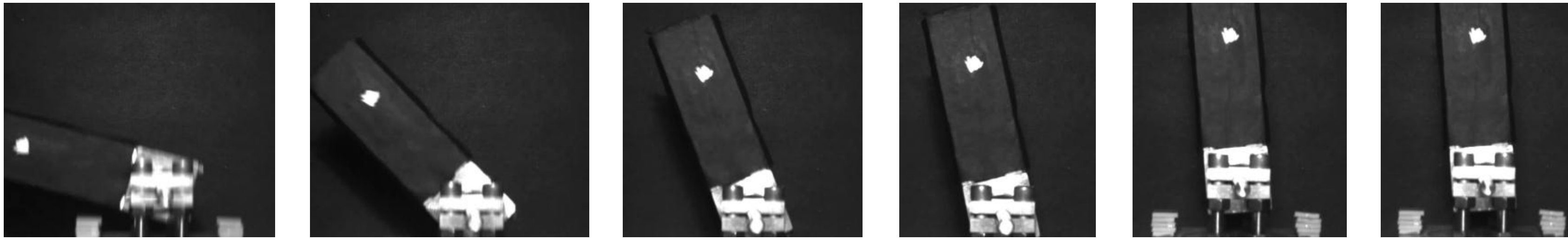


Stability of an Inverted Pendulum Under Nonsinusoidal Forcing

Samuel Shapero, Robert Hayward, Yiwei Cheng

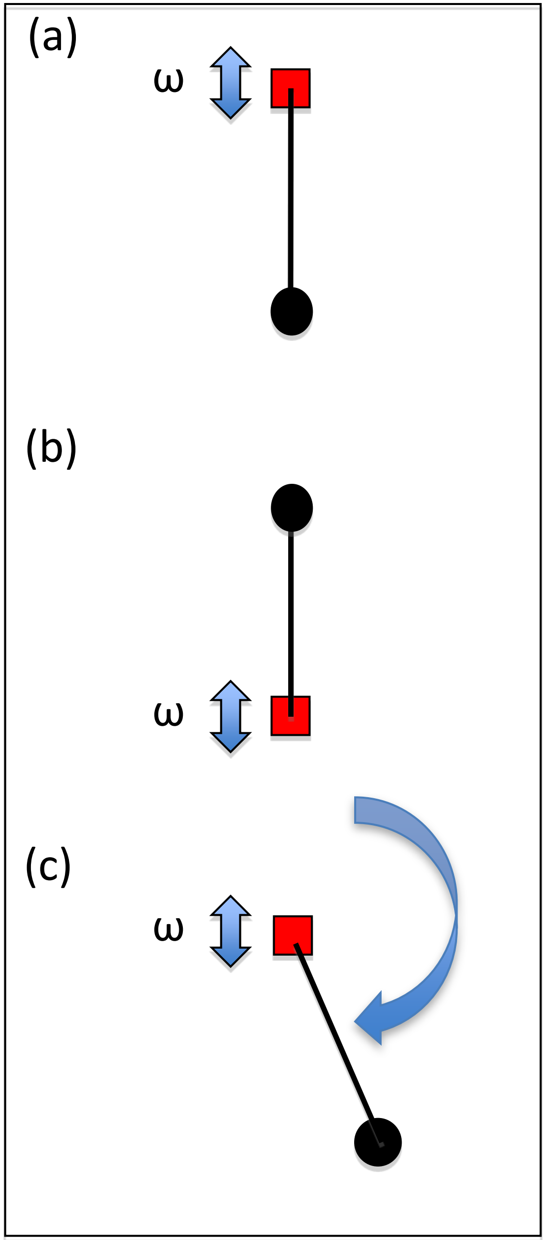
Part I – Introduction and Preliminary Simulation Results

Presented by: Yiwei Cheng

Contributions:

- Background Research
- Preliminary modeling and simulation
- Experimental data collection
- Website

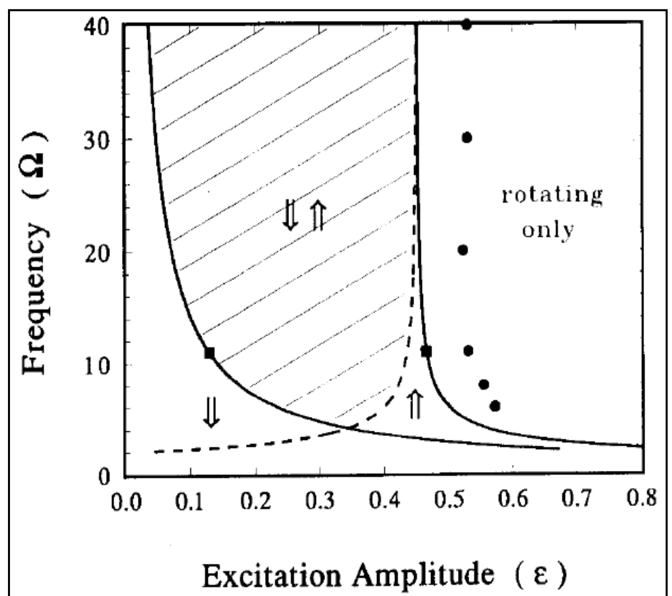
Introduction



The simple pendulum only has one stable state: the vertically down orientation (a)

But if the axis undergoes vertical perturbation, three different equilibrium states are possible:

(a) Stationary down
(b) Stationary up
(c) Continuous rotation in either direction



Regions of stability for each equilibrium state.

Blackburn et al., 1992.

Objectives

We investigated the dynamics of an inverted pendulum subjected to periodic forcing, both numerically and experimentally.

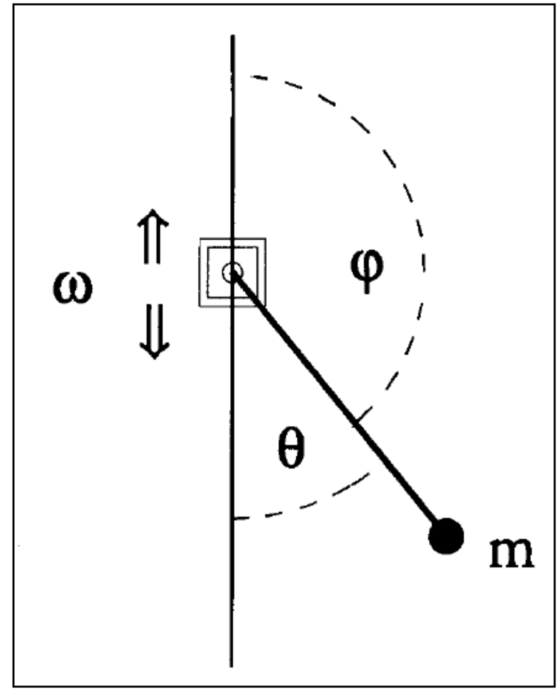
We were trying to answer the following questions:

1. What amplitudes and frequencies of periodic forcing allow the inverted pendulum to remain upright? (i.e. what are the regions of stability?)
2. Does the region of stability change when additional harmonics are added to the forcing function?
3. Can our theoretical findings be validated through laboratory experiments?

Modeling: Equation of Motion

General equation of motion for a pendulum subjected to vertical harmonic forcing

$$I\frac{\partial^2\theta}{\partial t^2}+b\frac{\partial\theta}{\partial t}+Mr[g-A\omega^2\cos(\omega t)]\sin\theta=0$$



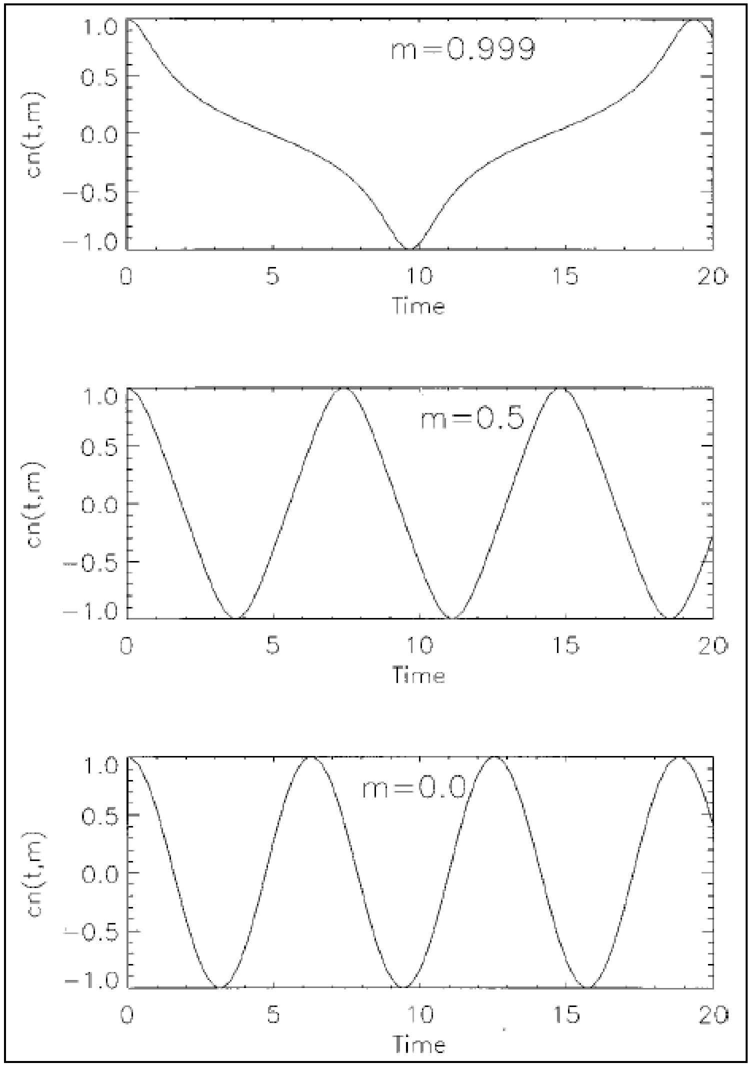
θ is the angular coordinate
b is the damping coefficient
I is the total moment of inertia of the system
M is the mass of the pendulum
Normalizing time according to the transformation ωt --> t

$$\frac{\partial^2\theta}{\partial t^2}+\left(\frac{1}{\Omega Q}\right)\frac{\partial\theta}{\partial t}+\left[\left(\frac{1}{\Omega^2}\right)-\left(\frac{A}{r}\frac{Mr^2}{I}\right)\cos(t)\right]\sin(\theta)=0$$

$Q = \omega_o I/b$; $\Omega = \omega/\omega_o$; $\omega_o=\sqrt{\frac{Mrg}{I}}$

$$\frac{\partial^2\theta}{\partial t^2}+\frac{1}{Q\cdot\Omega}\frac{\partial\theta}{\partial t}+\left[\frac{1}{\Omega^2}-\varepsilon\cos(t)\right]\sin(\theta)=0$$

Modeling: Jacobi Elliptic Function



Sanjuan, 1998

To create non-sinusoidal forcing we used the Jacobi Elliptic function ***cn***$(\omega t, m)$, where $m \in [0,1]$

Special cases:
$cn(\omega t, m=0) = \cos \omega t$
$cn(\omega t, m=1) = \operatorname{sech} \omega t$

Why Jacobi Ellipticals?

They mimic the dynamics of the inverted pendulum. As m → 1 the function has a period lengthening bottleneck around 0.

Modeling cont.

With the Jacobi Elliptical function, our equation of motion is now:

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{1}{Q \cdot \Omega}\frac{\partial \theta}{\partial t} + \left[\frac{1}{\Omega^2} - \varepsilon \cdot cn(t,m)\right]\sin(\theta) = 0$$

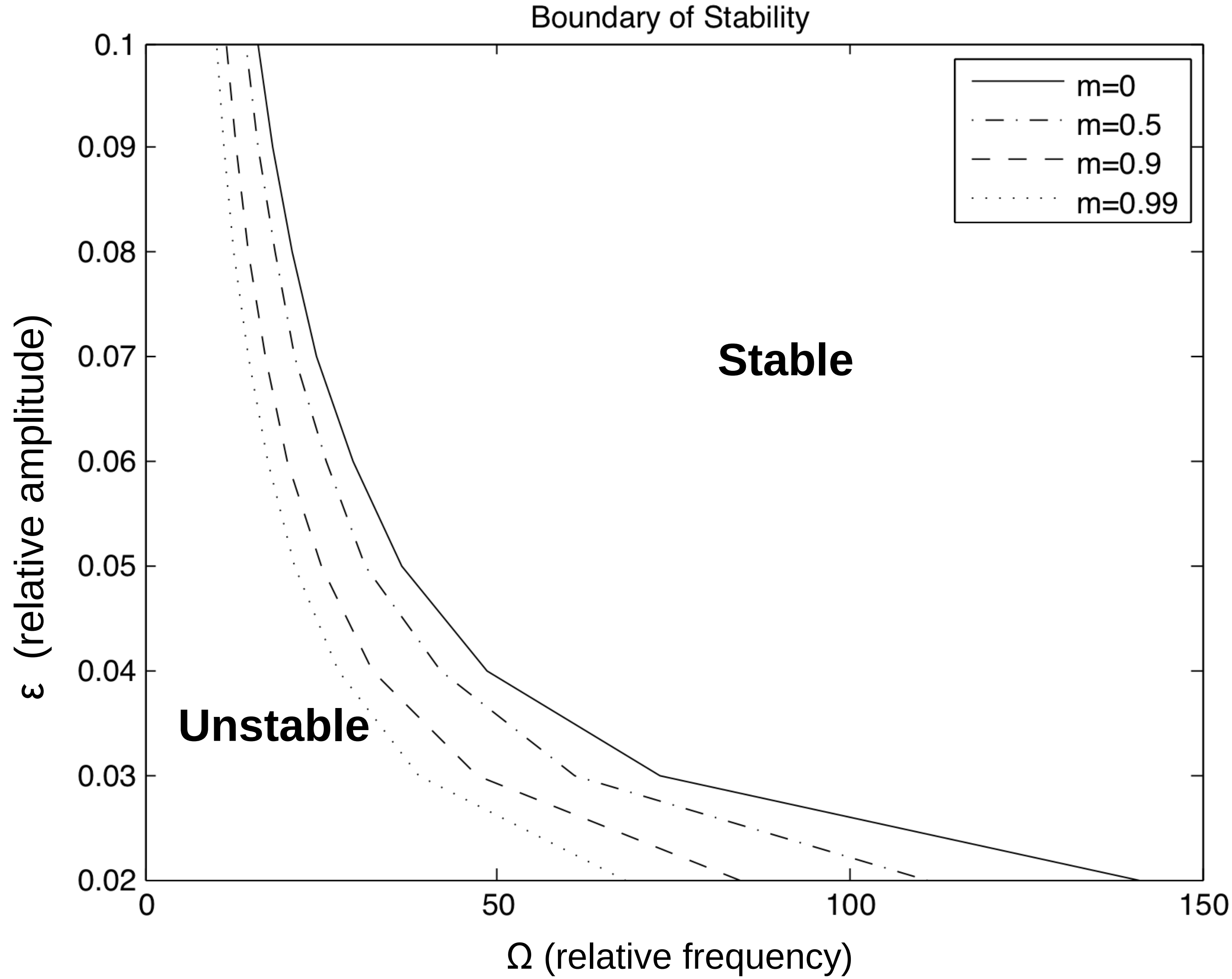
This is equivalent to the 2D system of differential equations:

$$\frac{\partial \upsilon}{\partial t} = -\frac{1}{Q \cdot \Omega}\upsilon - \left[\frac{1}{\Omega^2} - \varepsilon \cdot cn(t,m)\right]\sin(\theta)$$

$$\frac{\partial \theta}{\partial t} = \upsilon$$

We employed a fourth order Runge Kutta routine to compute numerical solutions to the equation of motion in MATLAB.

Preliminary Simulation Results



Initial Hypothesis: Region of stability increases as *m* increases.

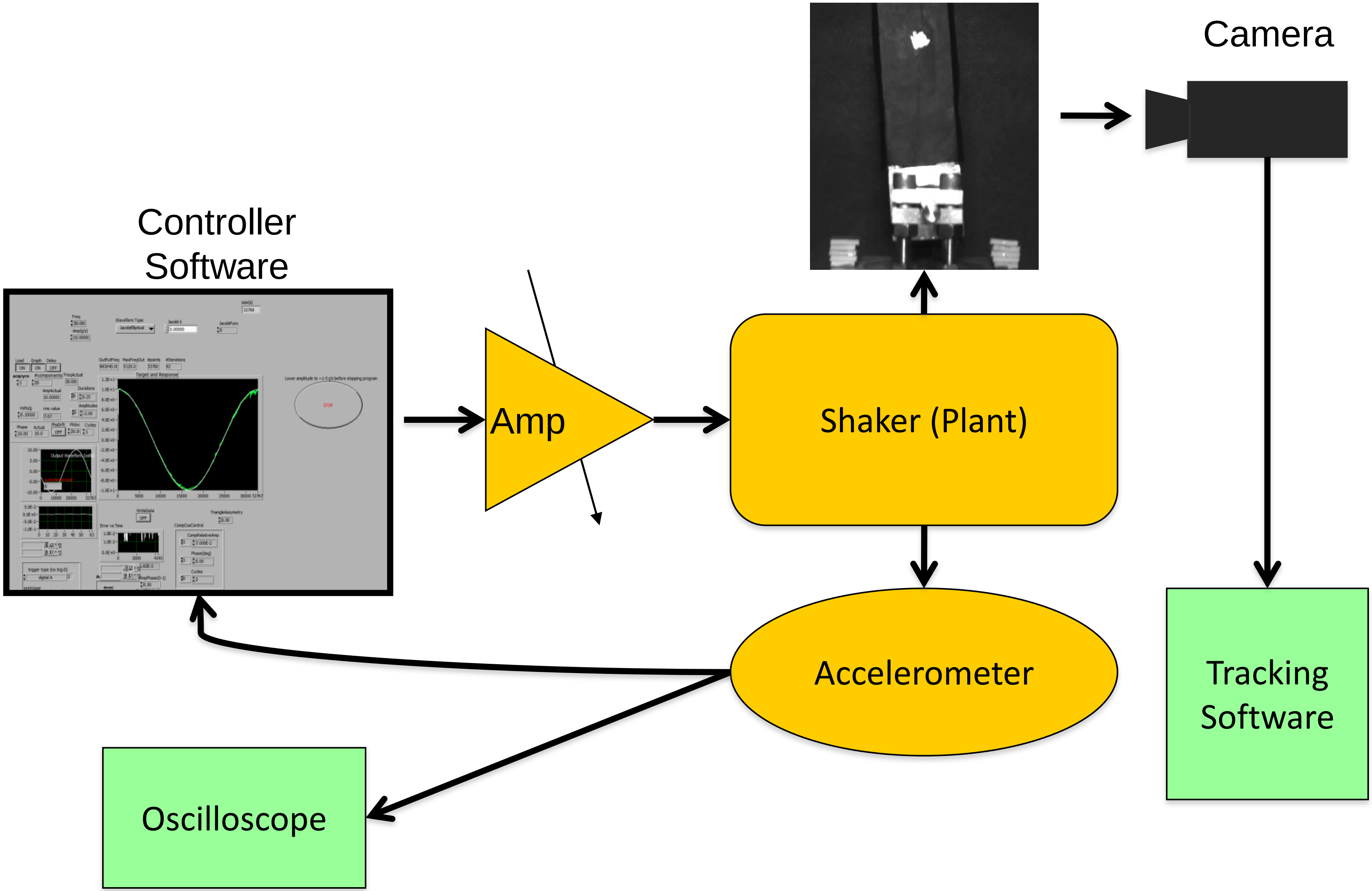
Part II – Experiments and Data

Presented by: Sam Shapero

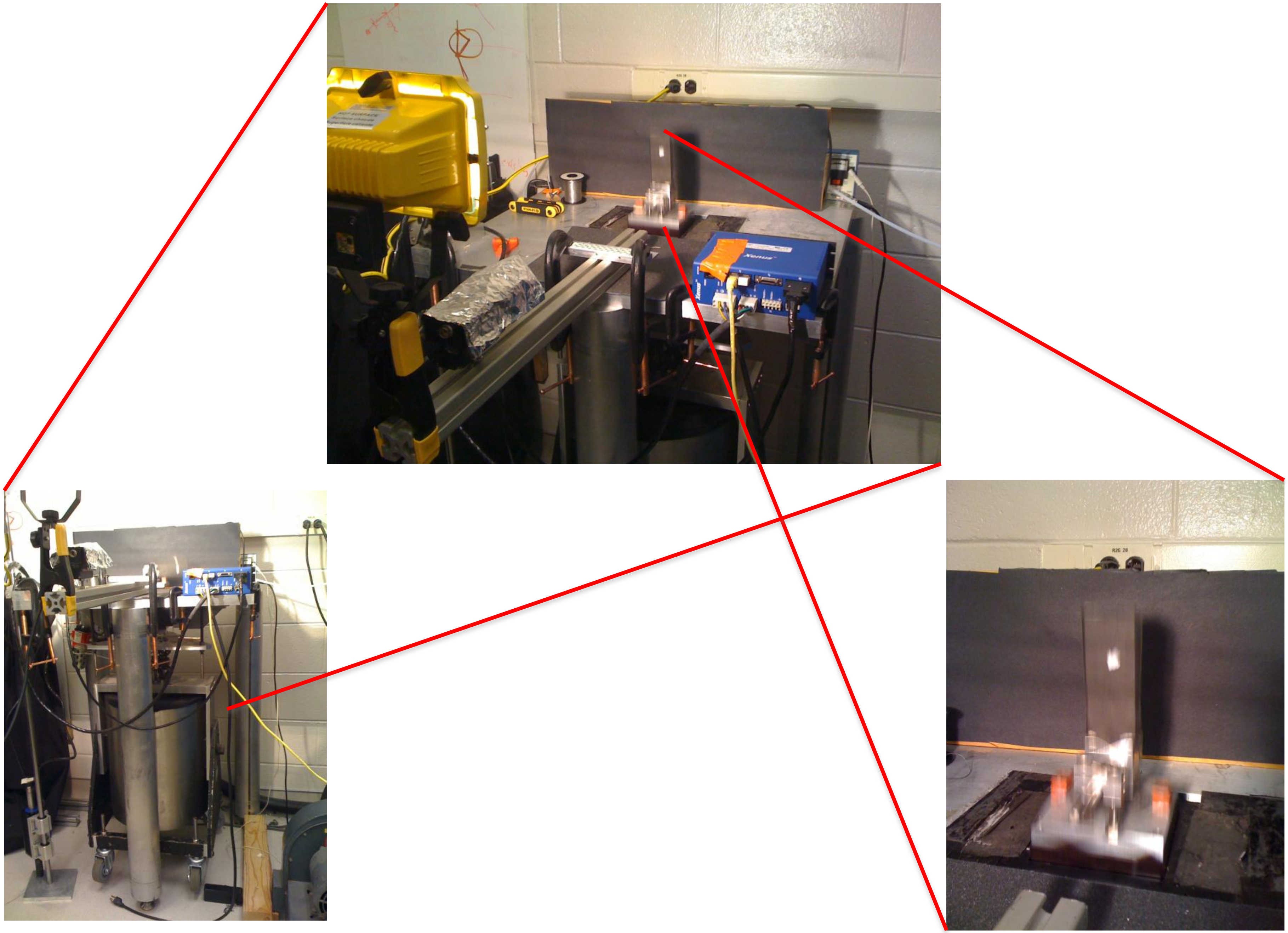
Contributions:

- Background Research
- Preliminary modeling and simulation
- Experimental data collection
- Data analysis

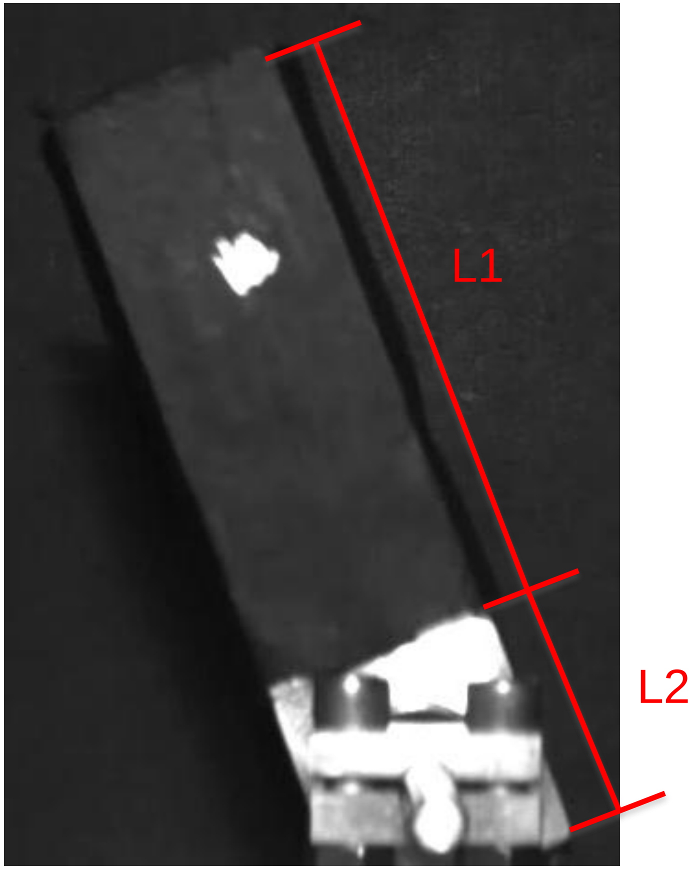
Signal Block Diagram



Experimental Setup



Experimental Parameters



$$\frac{\partial^2\theta}{\partial t^2}+\frac{1}{Q\cdot\Omega}\frac{\partial\theta}{\partial t}+\left[\frac{1}{\Omega^2}-\varepsilon\cdot cn(t,m)\right]\sin(\theta)=0$$

<u>Pendulum characteristics</u>

L1 = 8.1cm
L2 = 3.8 cm
r = 3.1cm
I/M = 23cm^2

$$\omega_o=\sqrt{\frac{Mrg}{I}}=11.5 rad/s$$

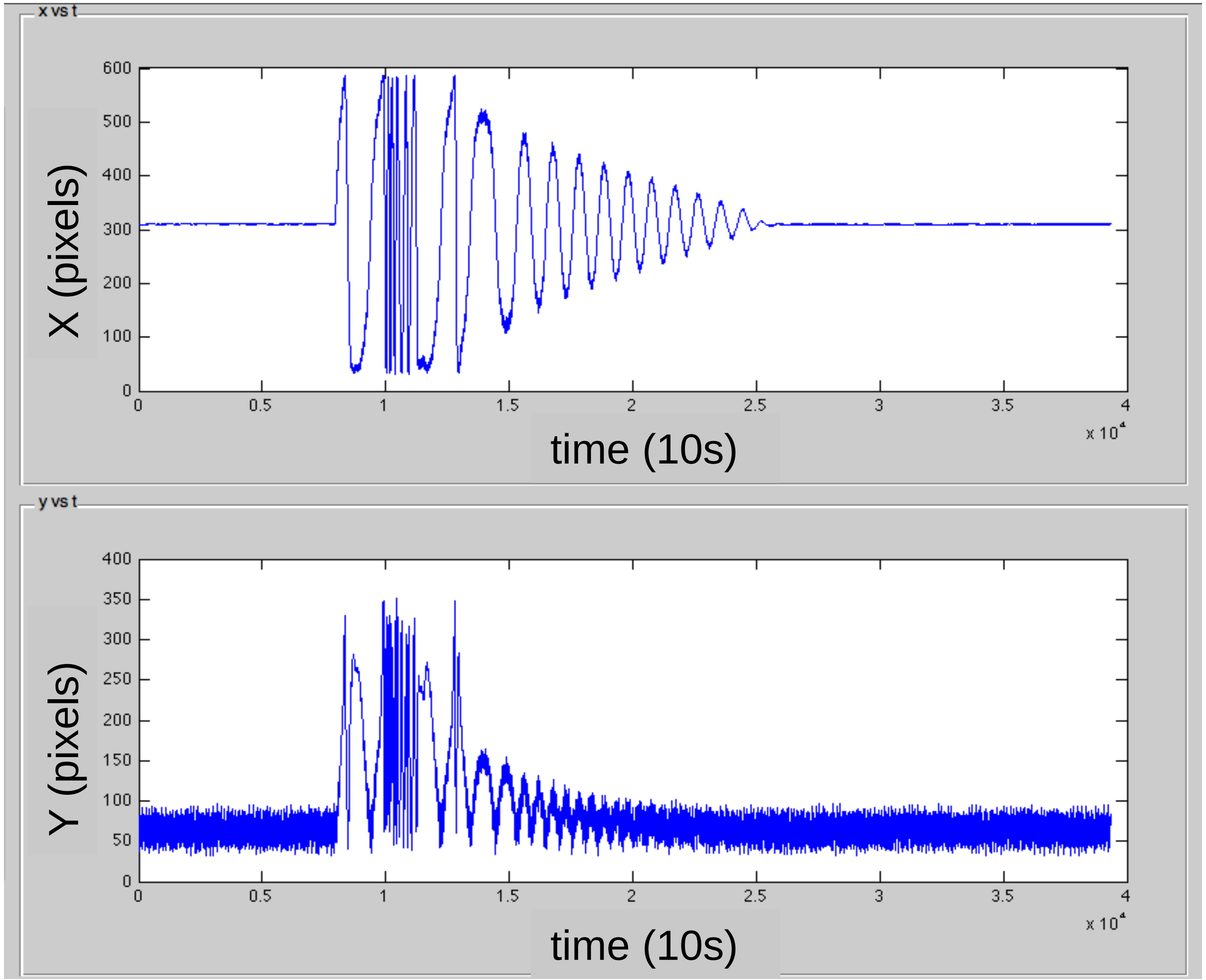
Experimental Procedure

We began by trying to stabilize the pendulum at a 26 Hz forcing frequency, increasing the acceleration until the pendulum was stable in the upright position

Once we'd found a stable point, we perturbed it, to measure the dynamics around the fixed point, including oscillation frequency and damping.

We then varied the driving frequency from 20 to 50Hz, and the eccentricity m from 0 to 0.999. At each node, we first attempted to stabilize the pendulum by setting the acceleration as high as the amplifier would safely allow, then decreasing the peak acceleration until the pendulum was no longer stable. We recorded this acceleration, as well as the vertical displacement of the shaker.

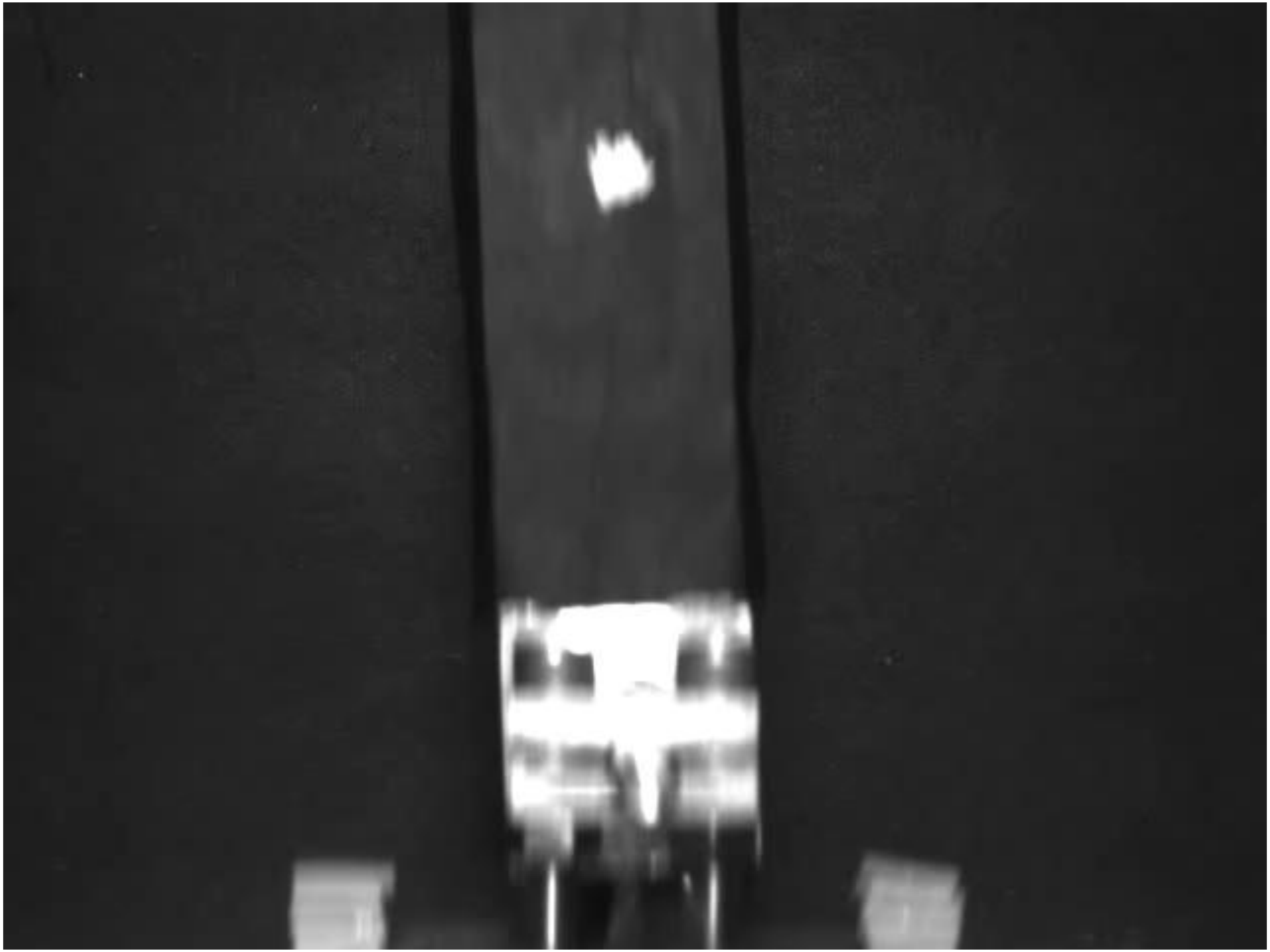
Tracking the Pendulum



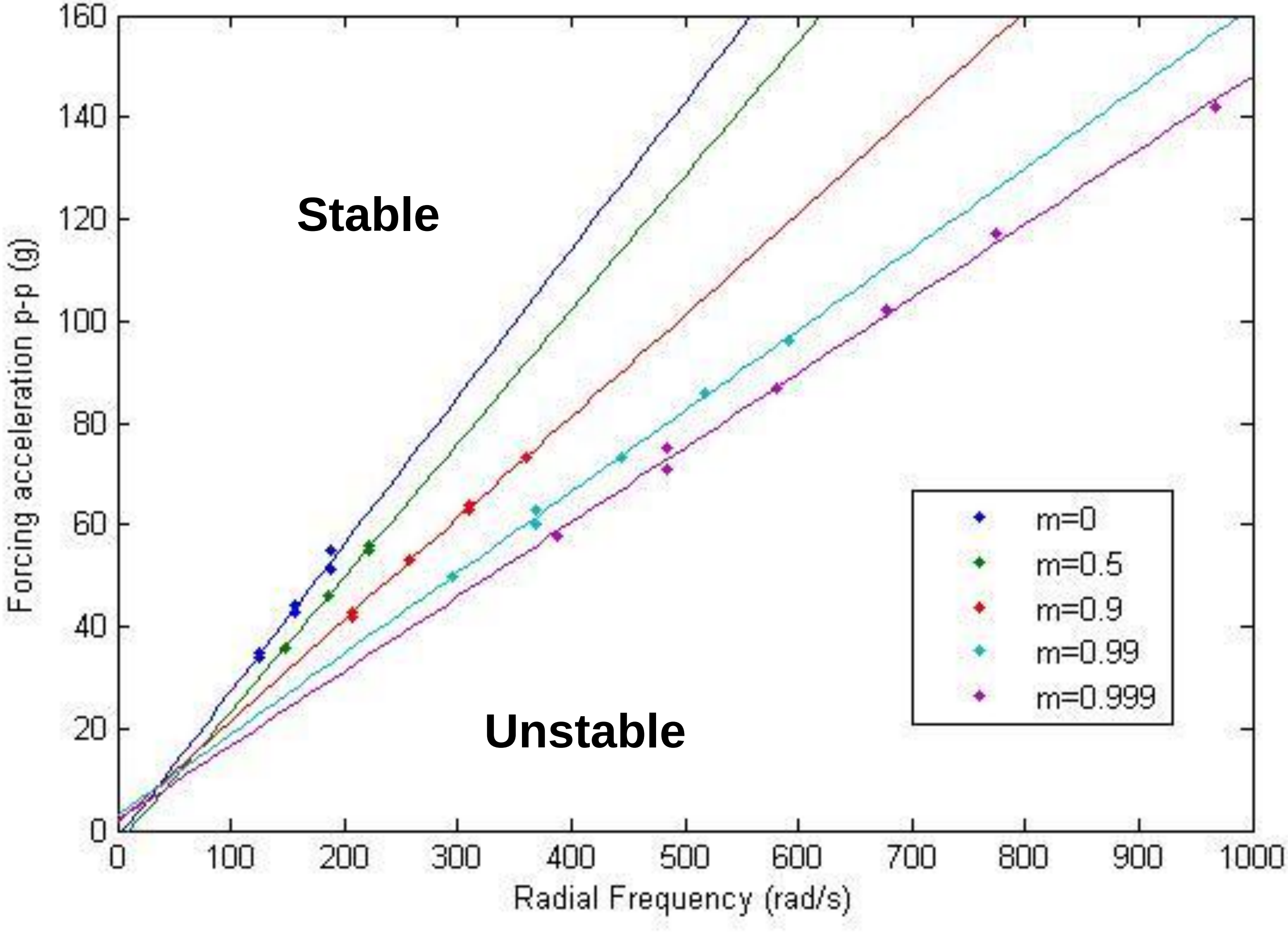
Inverted pendulum at 26Hz, m=0.999, 80g

Stable pendulum is perturbed, bounces off rails, then spontaneously regains stability.

Movie Slide

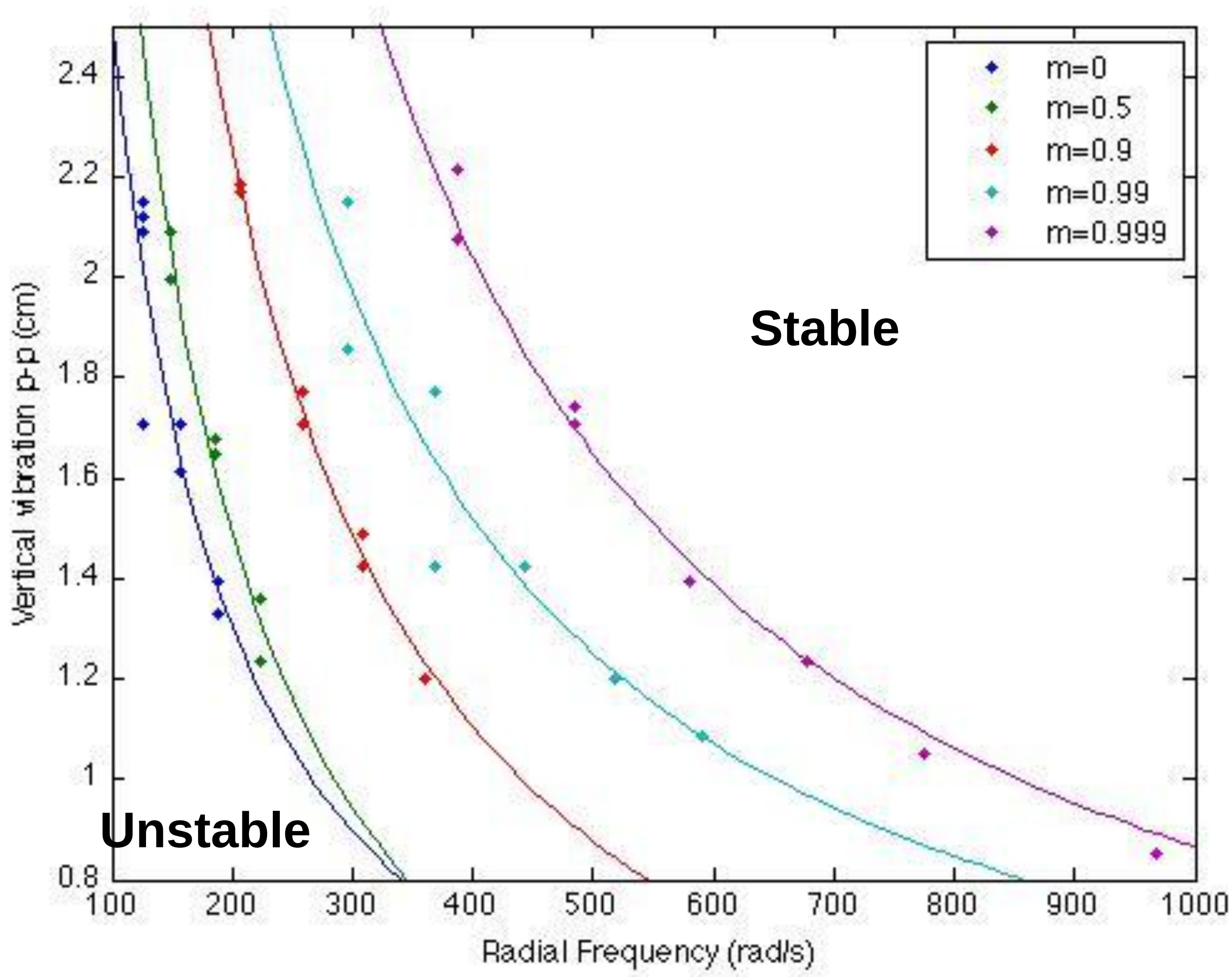


Acceleration vs. Frequency



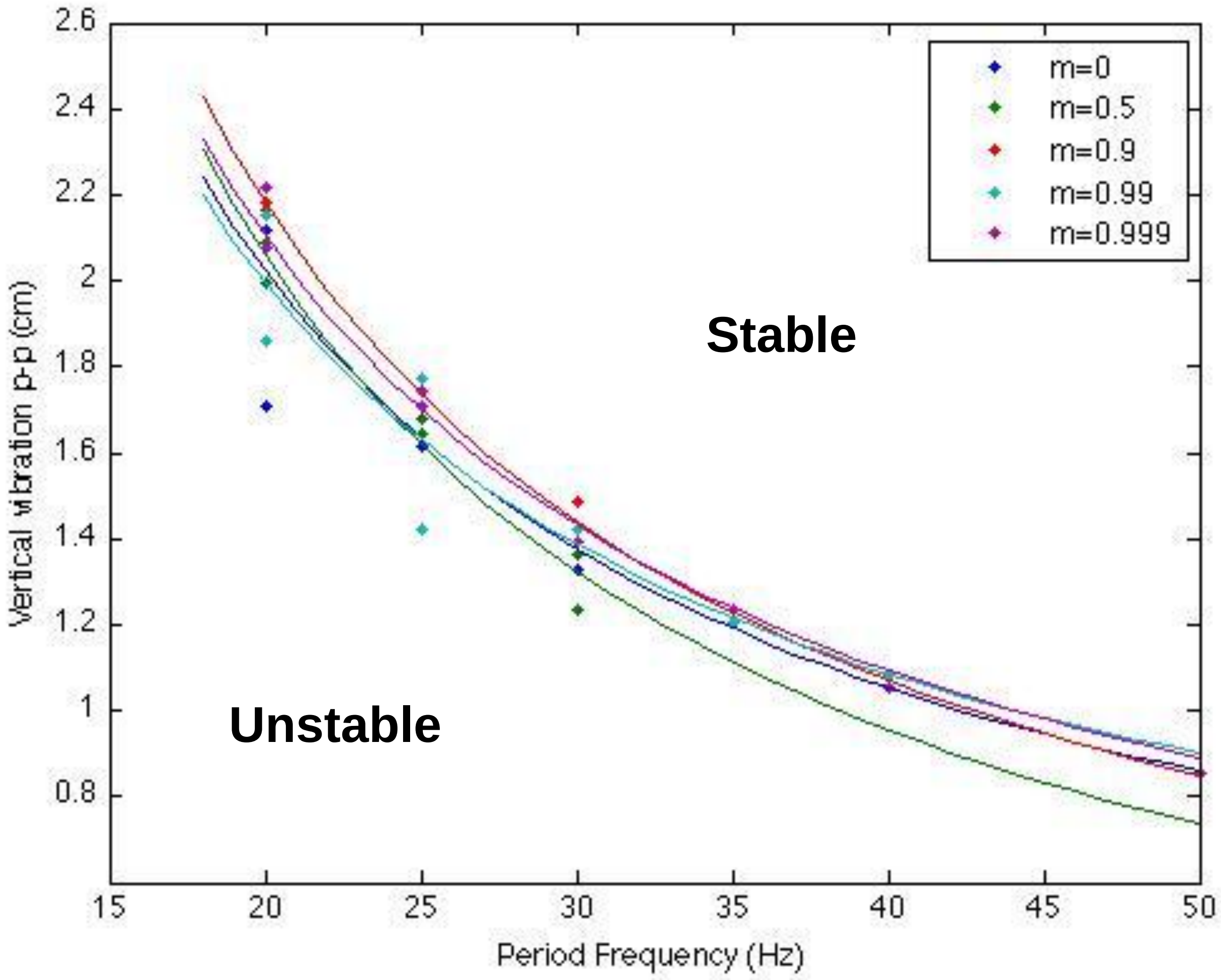
For each m, the acceleration required for stability is a linear function of the frequency

Displacement vs. Frequency



For each m, displacement had inverse relationship with frequency

Displacement vs. Periodic Frequency



Adjusting to periodic frequency eliminates any differences from eccentricity

Part III – Analysis

Presented by: Robert Hayward

Contributions:

- Background Research
- Modeling and Simulation
- Data Collection
- Analysis of Combined Numerical and Experimental Results

Experimental setup does not match initial model

- LabView controller enforces fixed period on the forcing function
    - cn(t,m) has period as a function of m
- We measure amplitude of displacement
    - The displacement amplitude of a body accelerating as cn(t,m) cannot be calculated trivially

Determine Proper Scaling/Stretching

We want the vertical displacement of the pivot, $f(t)$, to have the following properties:

- Should be 2π periodic
- $f(0)=1, f\left(\frac{\pi}{2}\right)=0, f(\pi)=-1, f\left(\frac{3\pi}{2}\right)=0$
- $\max\limits_{t\in[0,2\pi]} f(t)=1, \min\limits_{t\in[0,2\pi]} f(t)=-1$

First, fix the period

$$cn\left(\frac{T(m)}{2\pi}t, m\right)$$

where

$$T(m) = 4\int_0^{2\pi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

has period 2π

Now find scaling

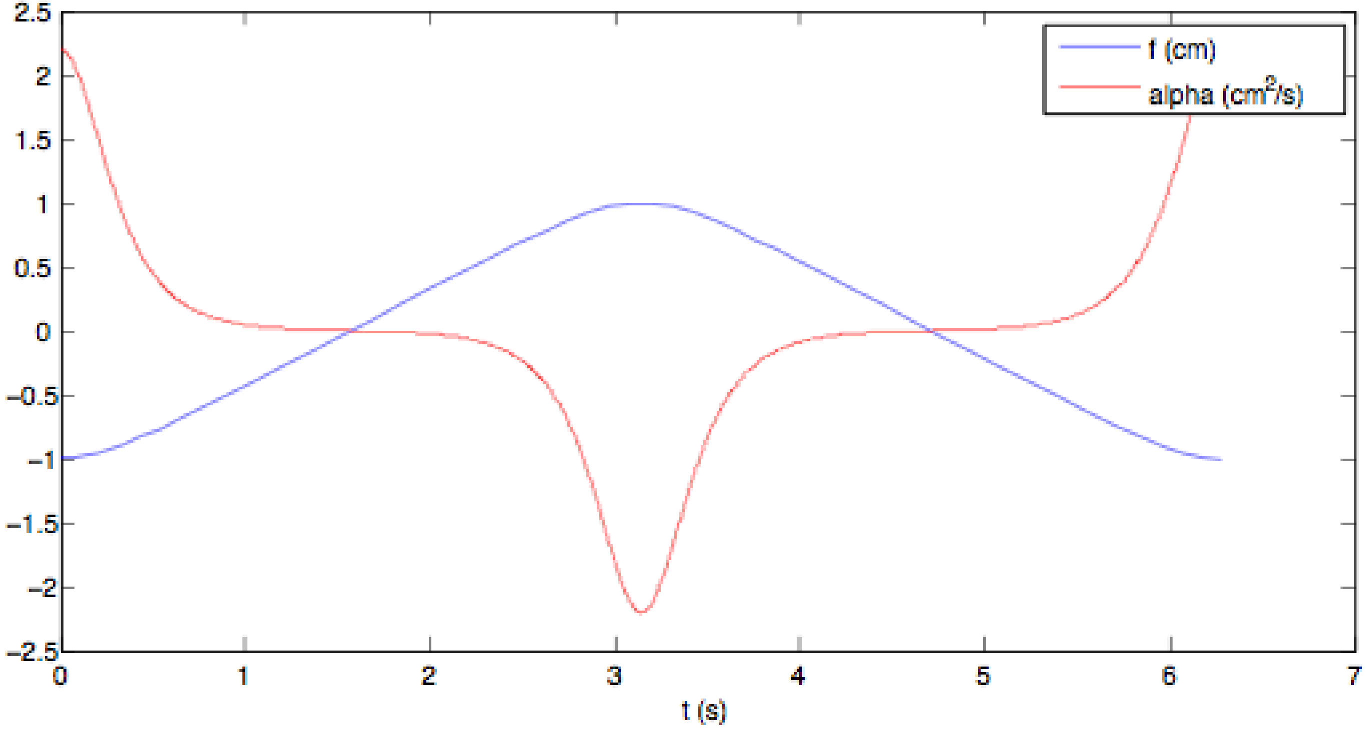
$$\alpha(t,m) = \rho(m)cn\left(\frac{T(m)}{2\pi}t, m\right)$$

where

$$\rho(m) = \frac{1}{\int_0^{\pi/2}\int_0^{t} cn\left(\frac{T(m)}{2\pi}u, m\right)du\,dt}$$

creates unit-amplitude displacement

End Result for m=0.9999



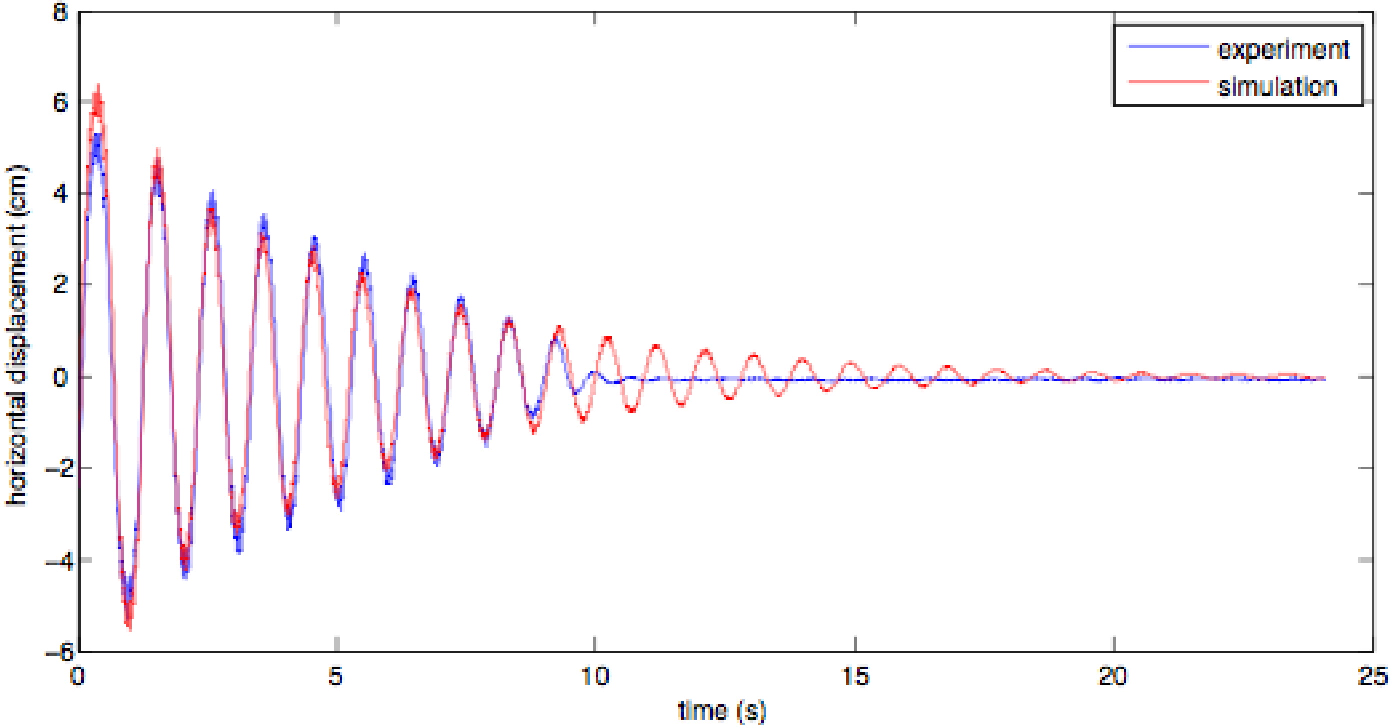
Revised Model

With the scaled and stretched Jacobi Elliptical function, our equation of motion is now:

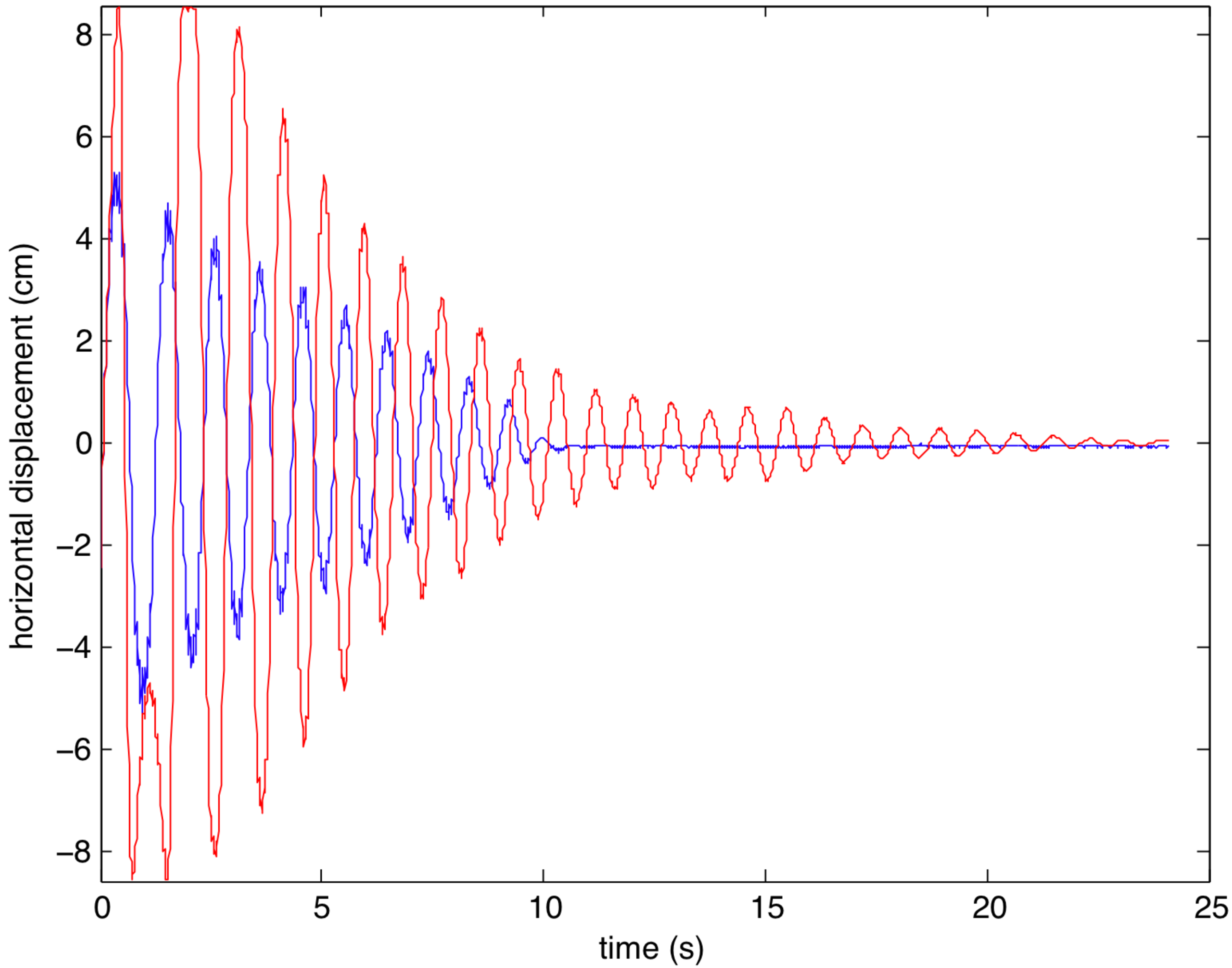
$$\frac{\partial^2\theta}{\partial t^2}+\frac{1}{Q\cdot\,\Omega}\frac{\partial\theta}{\partial t}+\left[\frac{1}{\Omega^2}-\varepsilon\cdot\;\alpha(t,m)\right]\sin(\theta)=0$$

This is consistent with Blackburn for m=0.

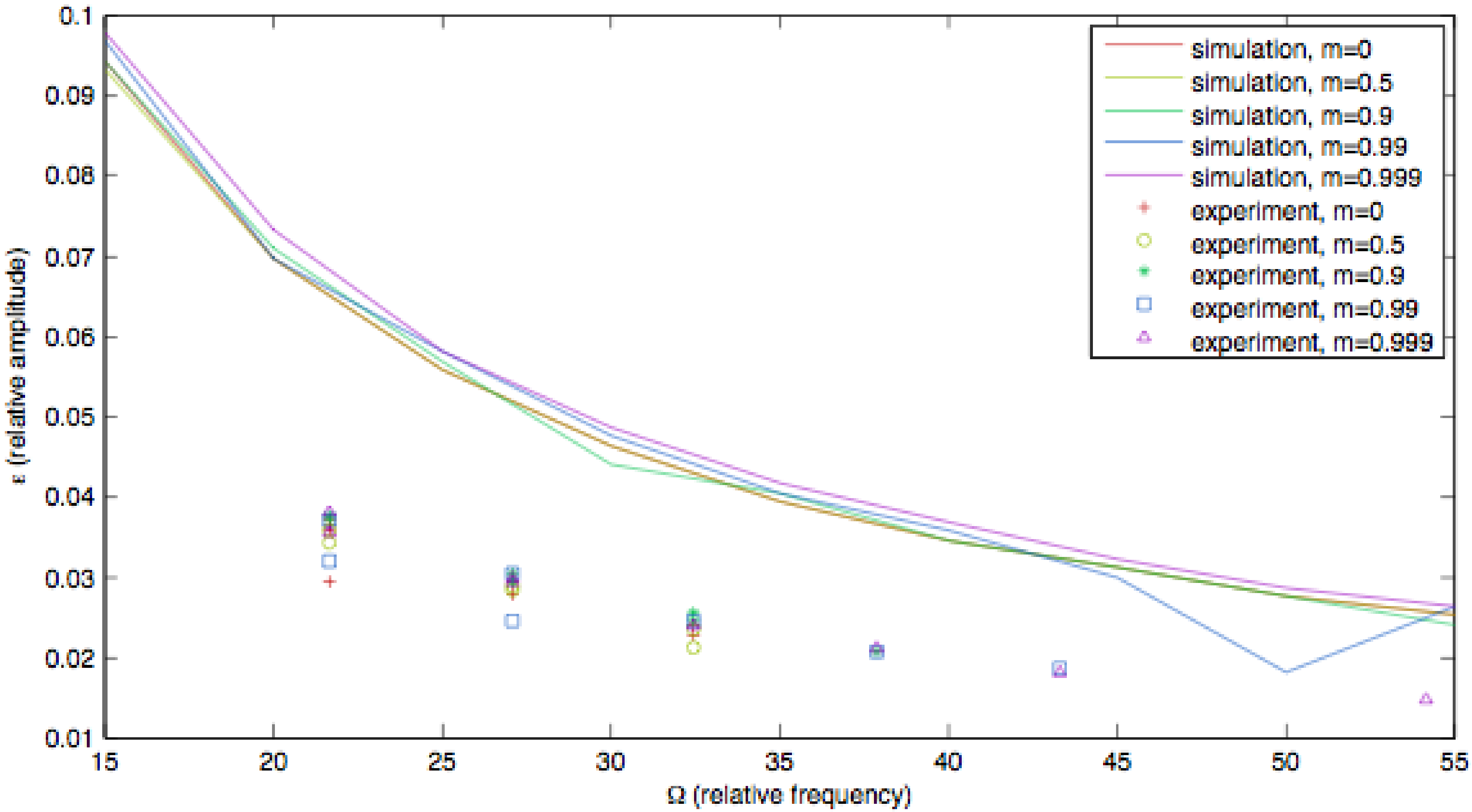
Now, fit numerical parameters to match experiment



$$Q = 11, \omega_0 = 5.8 s^{-1}$$


horizontal displacement (cm)
8
6
4
2
0
−2
−4
−6
−8
0
5
10
15
20
25
time (s)

New stability diagram



Caveats

- Search for boundary of stability lacks robustness
    - Only considers one initial condition for each point
    - Integrates for an arbitrary time (1000 periods of forcing function)
    - Does not quite reach the “true” edge of stability
- Damping does not seem to match qualitatively between model and experiment
- Numerical results seem to be more sensitive (with regard to stability) than experimental results.
- Experimental stability criterion is somewhat subjective

Conclusion

We were able to determine the regions of stability of the forced inverted pendulum, with results similar to Blackburn et al, 1992.

Varying m in the Jacobi elliptical did not significantly alter the region of stability of the inverted pendulum.

This was indicated by both numerical and experimental results.

Future Work

Examine the basin of attraction experimentally (requires more complicated apparatus).

Use the Jacobi elliptical function in displacement instead of acceleration function.

Acknowledgements

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References

J. A. Blackburn, H. J. T. Smith, and N. Gronbechjensen, Stability and Hopf Bifurcations in an Inverted Pendulum, *American Journal of Physics* **60**, 903 (1992).

M. A. F. Sanjuan, Using Nonharmonic Forcing to Switch the Periodicity in Nonlinear Systems, *Physical Review E* **58**, 4377 (1998).

Resources

Website: http://invertpend.wordpress.com/