Plinko Dynamics

A Story of Pinball Chaos as Told by Ski Slope Chaos

Andrew Hardin - Andrew Masse - Christopher Cordell

8 December 2011

Introduction & Background

Andrew Hardin

Project Overview & Background

- **Original Observation**
  Plinko demonstrates chaos as evidenced by a sensitivity to the initial conditions (i.e., trajectories diverge from similar starting points)

- **Original Hypothesis**
  The sensitivity to initial conditions is evidence of rich, chaotic dynamics that can be well-understood by studying the basins of attraction resident within the system and by treating each lattice point as a compact physical system for examining the dynamics of an arbitrarily large Plinko board to look for long-term periodic (or aperiodic) behavior and structure

CBS/Mark Goodson Productions.

http://www.youtube.com/watch?v=uz8b3_jXeek

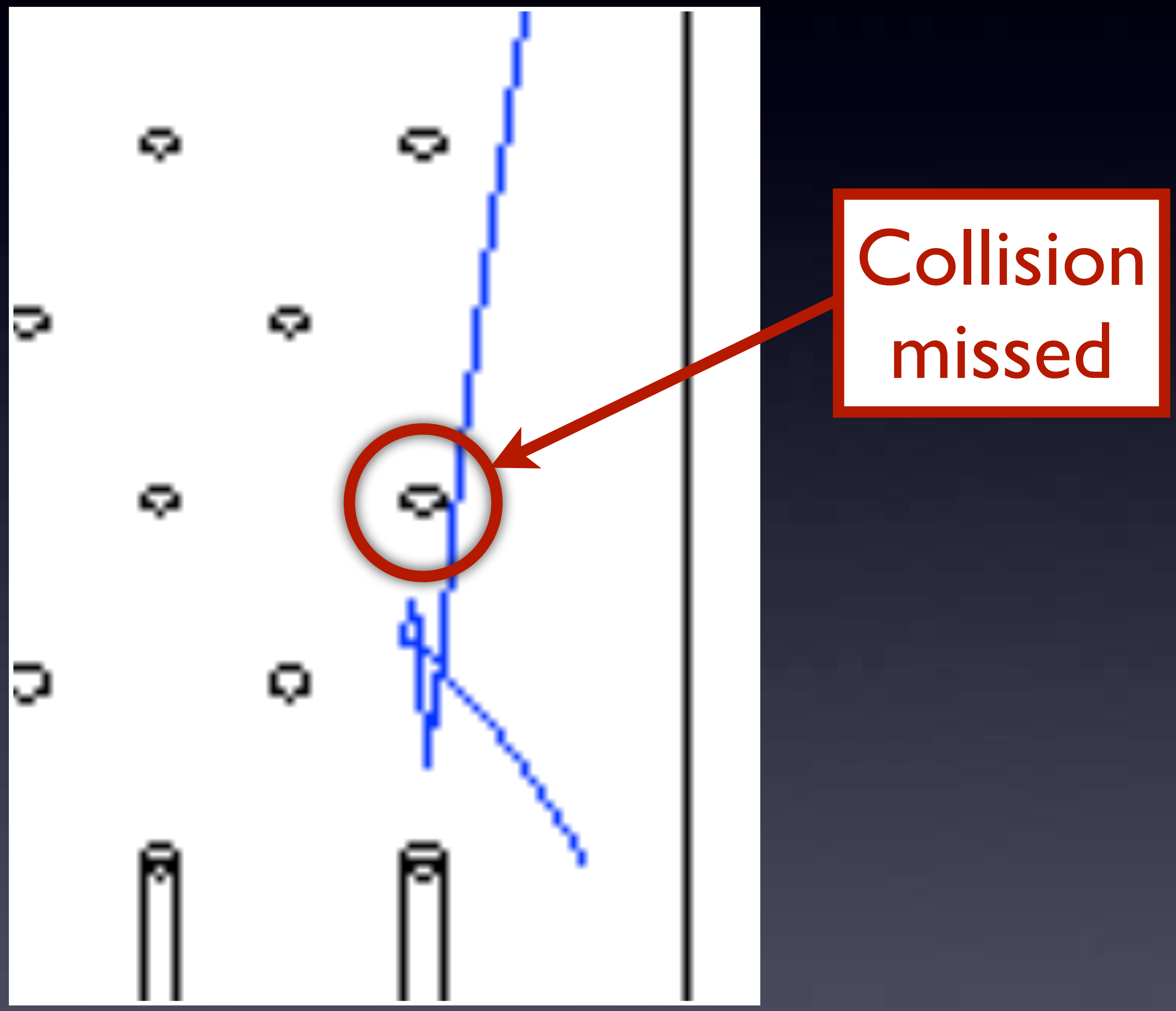
Plinko on The Price Is Right

Pinball & Billiard Chaos

- Boltzmann - statistical mechanics, ergodic theory and ergodic hypothesis on fluids of hard spheres
- Sinai, Y. G. "Dynamical systems with elastic reflections." *Russian Mathematical Surveys*. 1970 - ergodic properties of dispersing billiards
- Arroyo, Markarian, and Sanders. "Bifurcations of periodic and chaotic attractors in pinball billiards with focusing boundaries." *Nonlinearity*. 2009 - modification to the elastic collision rule that reduces energy in the system to show chaotic properties of non-conservative dynamics (phase space contraction, strange attractors, etc)
- Pring and Budd. "The dynamics of a simplified pinball machine." *IMA Journal of Applied Mathematics*. 2011- Pinball machine with a modified collision rule that injects energy into the system using "active" oscillators

The Problem With Pinball Chaos

- **Difficult to model puck behavior**
  - Solve multiple simple ODE's?
  - Closed-form solve using parabolas?
  - Solve a single complex ODE with each pin represented as a "force field"?
- **Difficult to detect collisions**
  - MATLAB "events" function in ODE solver for discrete collisions?
  - Quartic/root solver for parabolas?
  - How to model pegs using force fields while disallowing crossover?



MATLAB collision detection was unreliable

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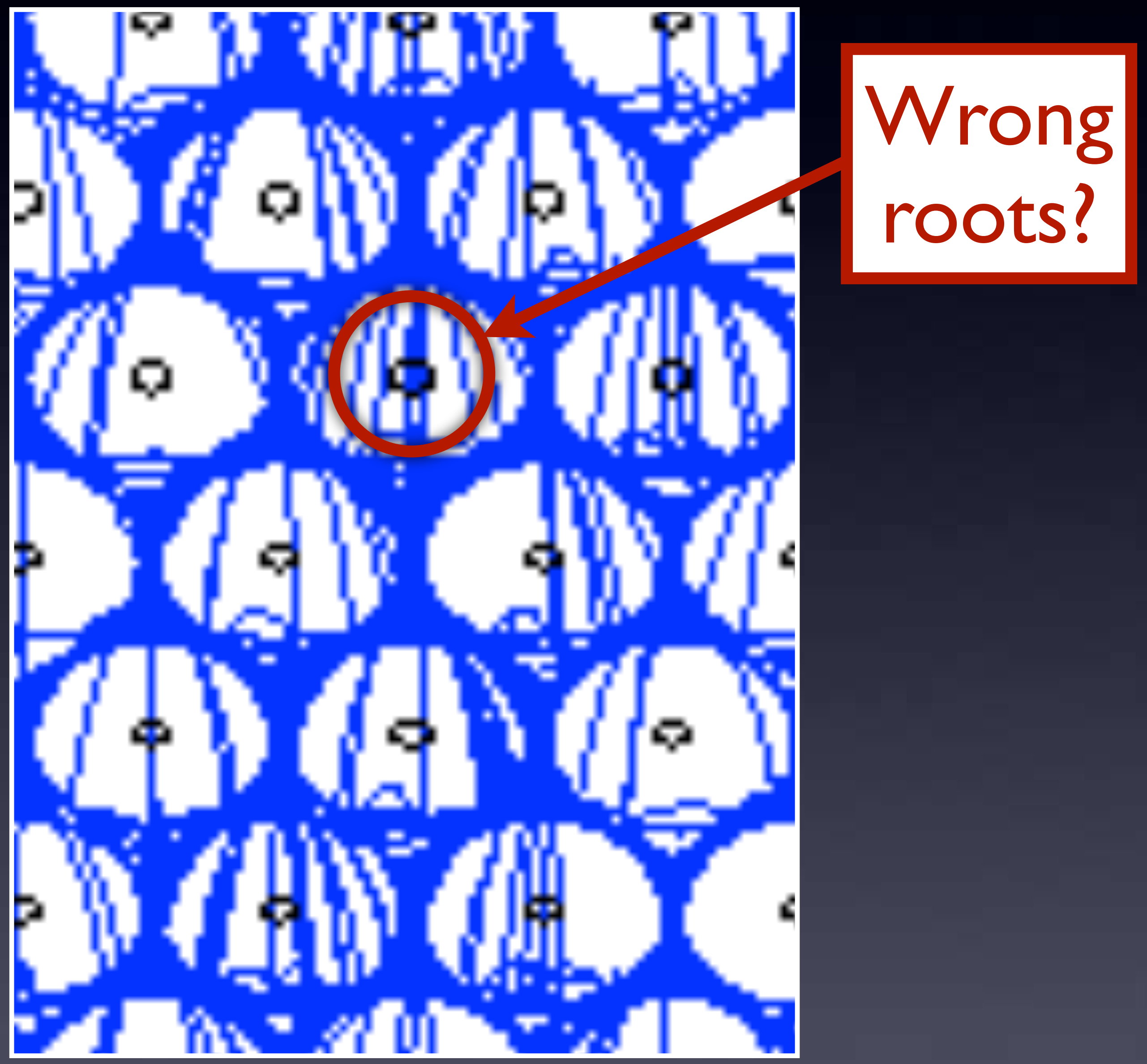
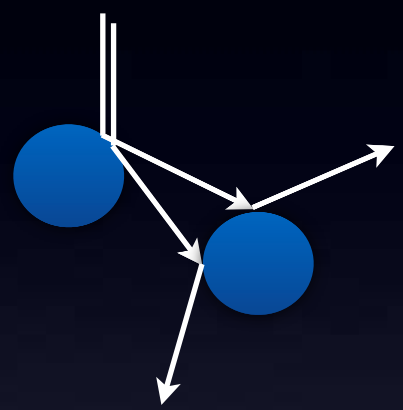


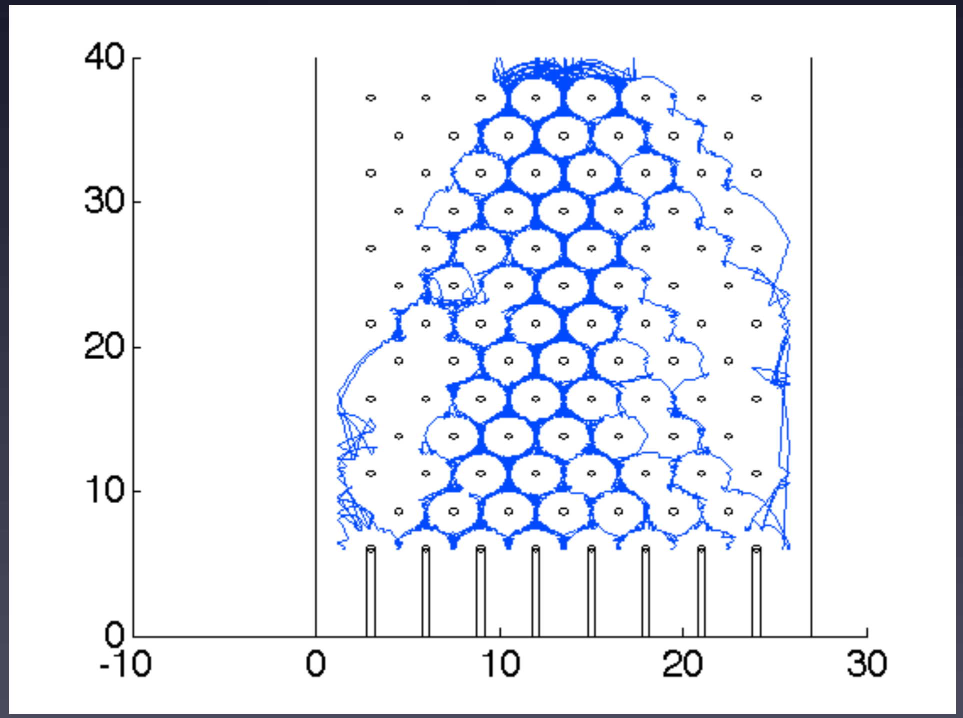
Even root-solving for collisions was difficult

The Problem With Pinball Chaos

- **Exponential divergence, no attractor**
  - Trajectories diverge exponentially with each collision
  - There is no "attractor" because the system is not dissipative
  - Lower coefficients of restitution help with divergence, but the model could not handle nearly inelastic collisions
- **No emergence of basins**
  - Without an attractor, the map just looks like noise



Notional example of diverging trajectories



Parabolic trajectories, root solving collisions, 80% restitution

The Problem With Pinball Chaos

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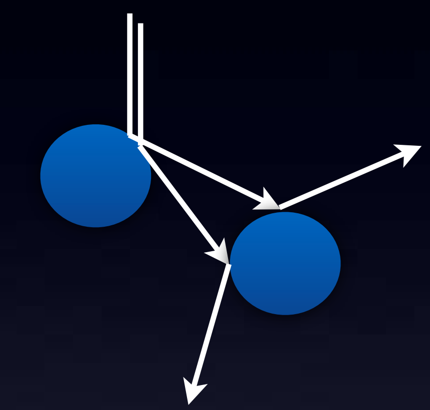

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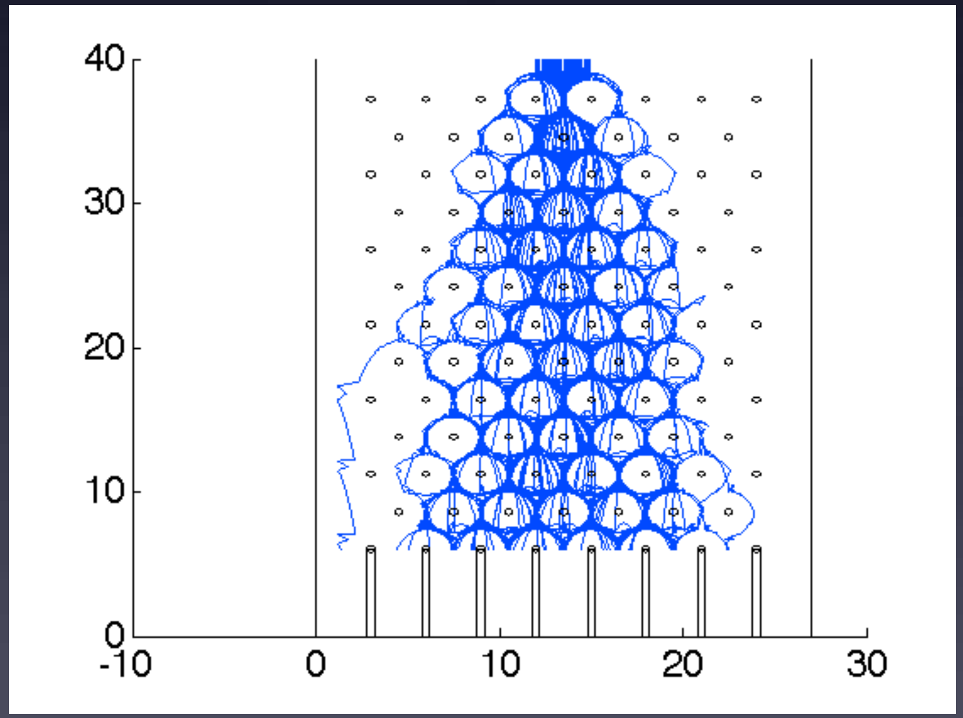


Parabolic trajectories, root solving collisions, 50% restitution

Ski Slope Chaos

- Detailed by Edward Lorenz in The Essence of Chaos

- Computer simulation and analysis of the trajectory of a particle on a ski slope with moguls

- Particle is acted on only by gravity, friction and the normal force generated between the particle and the mogul field

- The mogul peaks repel the particle while the depressions between moguls attract the particle

Plinko as Ski Slope Chaos

- Instead of discrete collisions, the system was modified to be a continuous force model representing the pins as “force fields”
- This modification led us to re-tool the experiment as an investigation of ski slope chaos using magnets instead of moguls
- The use of magnets allows for re-using the existing simulation with only slight modifications to the equations of motion
- Magnets (as opposed to moguls) also allows easier physical experimentation, something that the Lorenz results lack

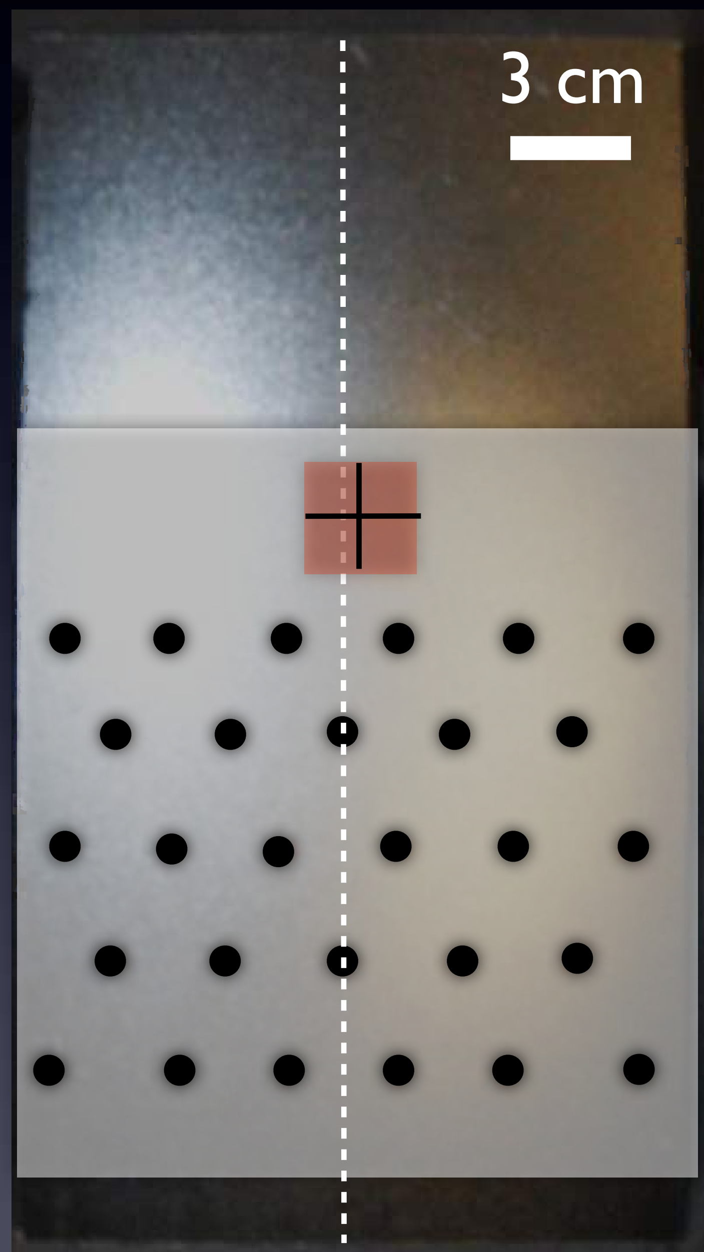
Investigative Procedure

- Construct the simulation using a simplified model of the dynamics and assumed parameter values
- Experimentally determine parameter values to calibrate the simulation to the experiment
- Validate the simulation with experimental data to show that the model accurately captures the dynamics of the system
- Explore the system's dynamics with the simulation to reconstruct the attractor and map the basins of attraction

Experimental Setup and Data Acquisition

Andrew Masse

Building a Magnetic Plinko Slope



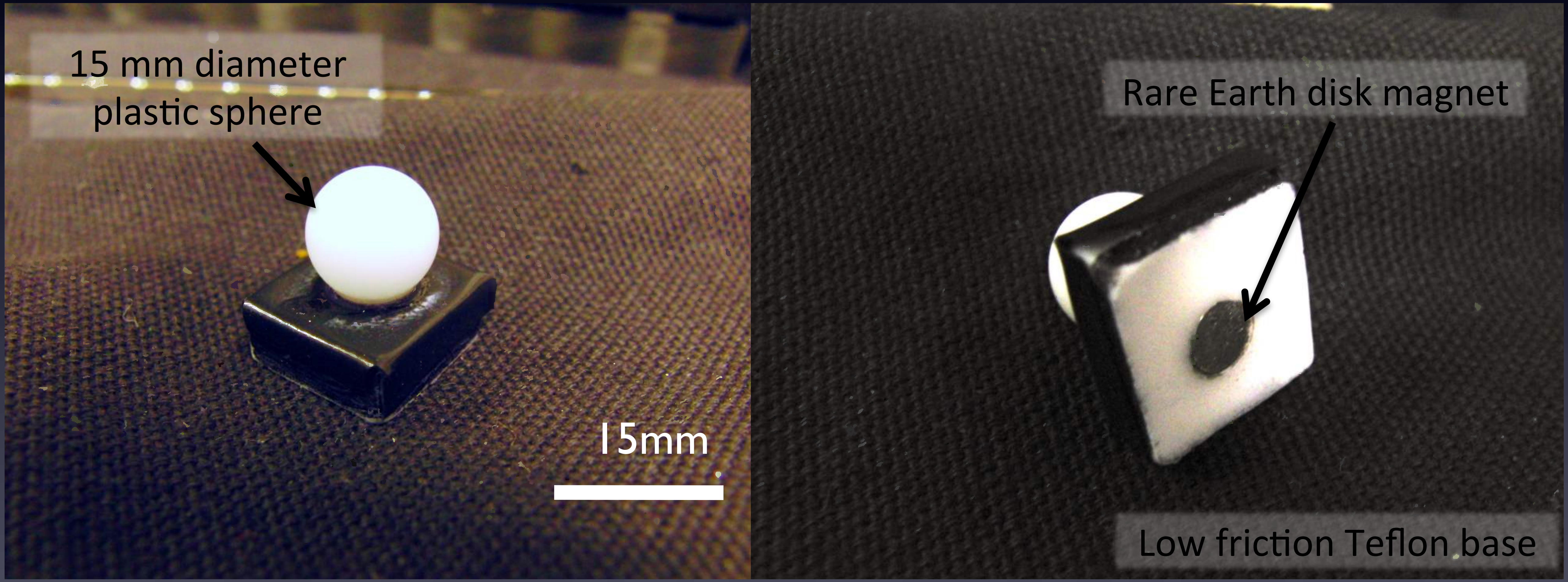
- **Materials**
  - Low coefficient of friction
  - Surface must be smooth and uniform
  - Buffer layer to avoid attracting events (effectively decreasing the magnitude of the magnetic force)
- Magnets
  - Used rare Earth magnets
  - attracting configuration

Starting region

Region in video frame

Magnetic Plinko Puck

- Coefficient of friction ≈ 0.15

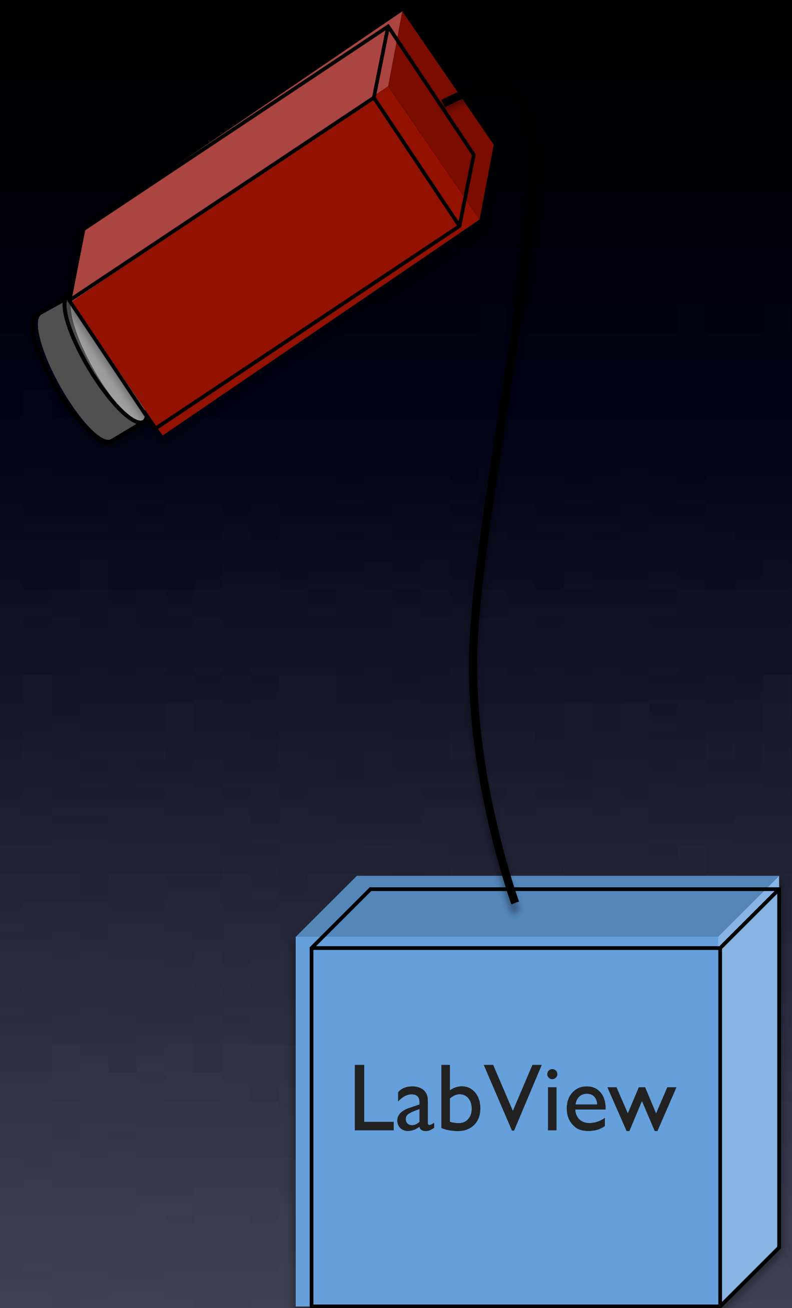


Full Setup

- 100 runs taken in each quarter of the red starting region

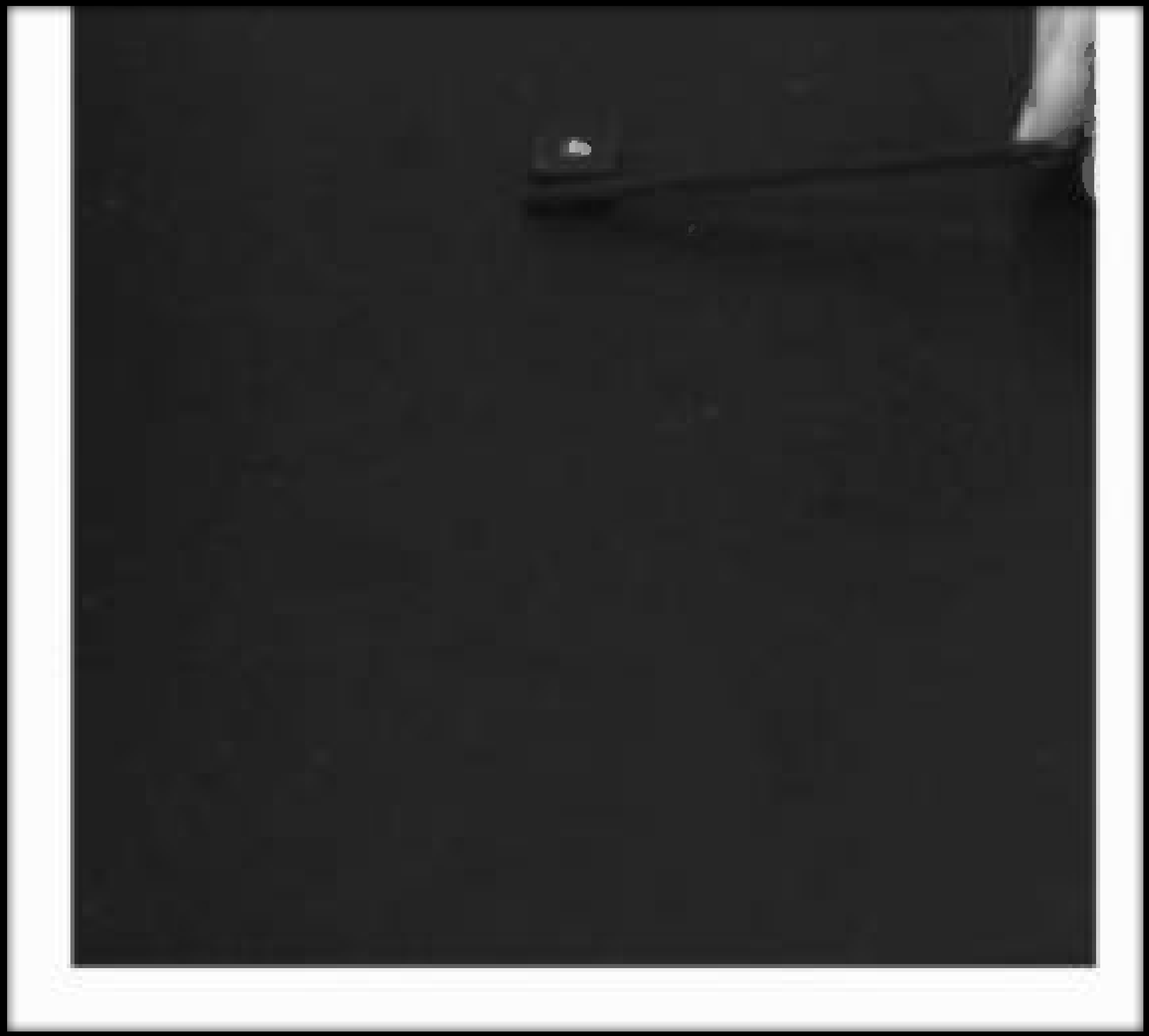
slope angle $\theta \approx 18^{\circ}$

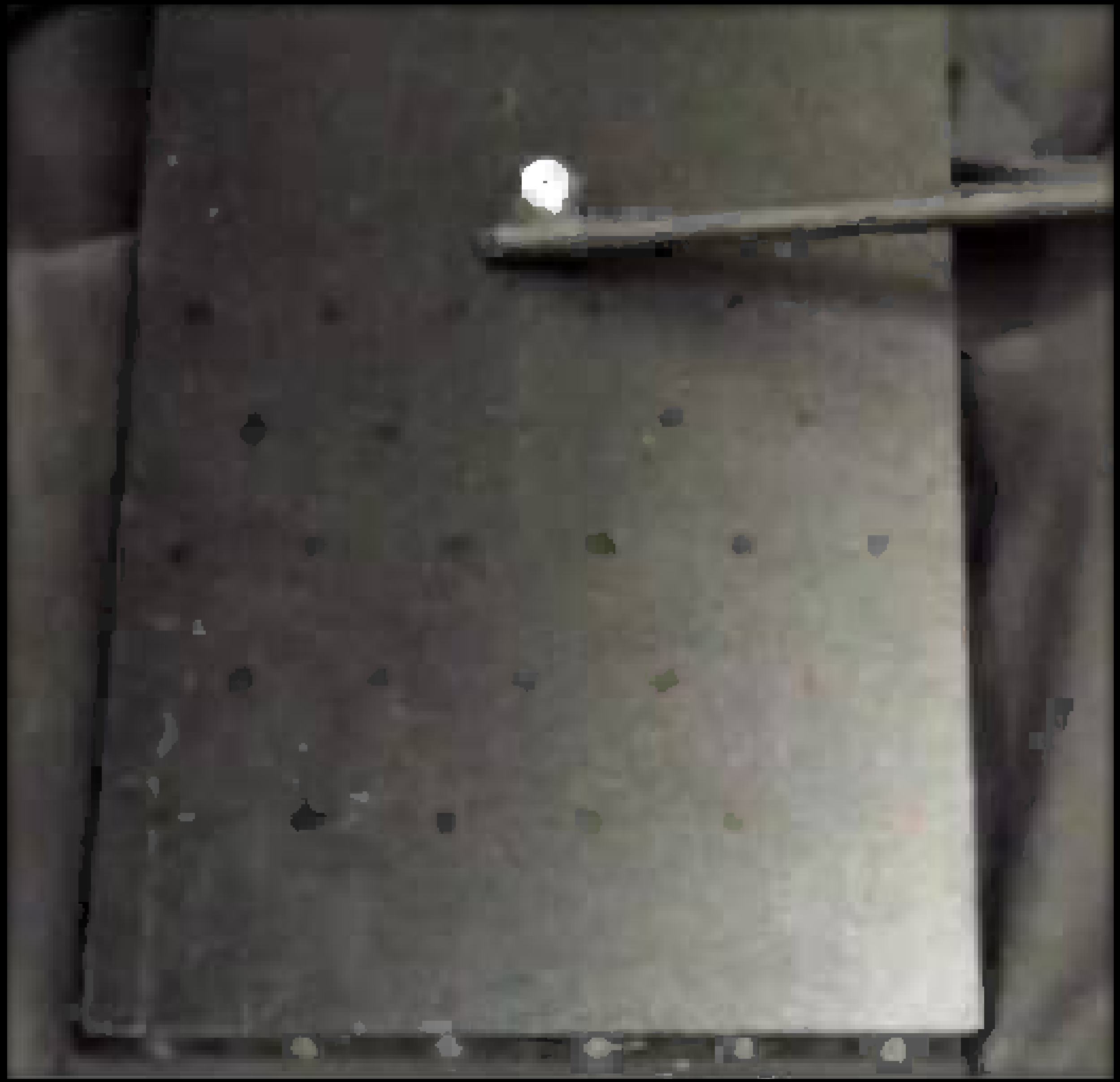
Allied Vision PIKE Camera

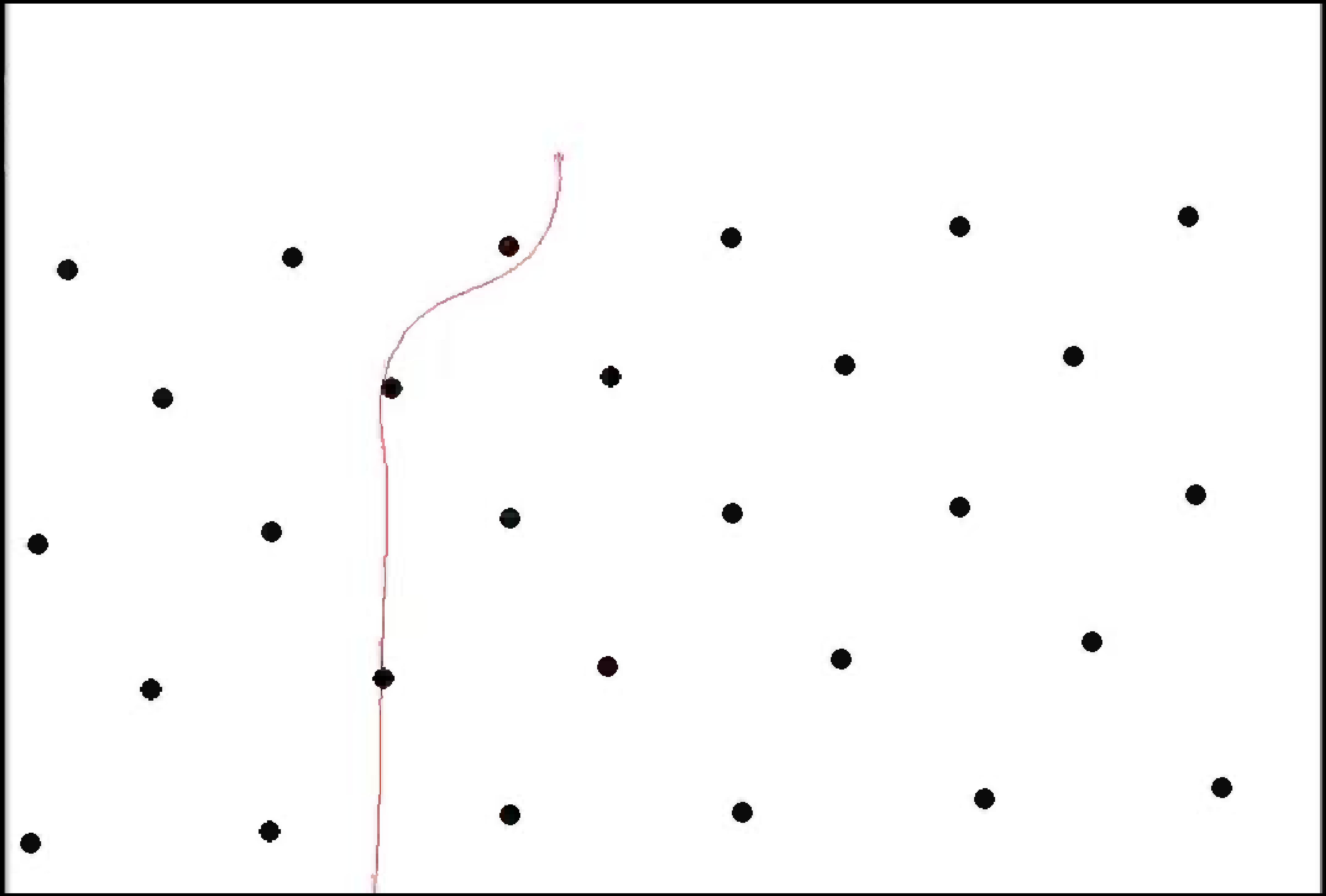


real time tracking for rapid data collection

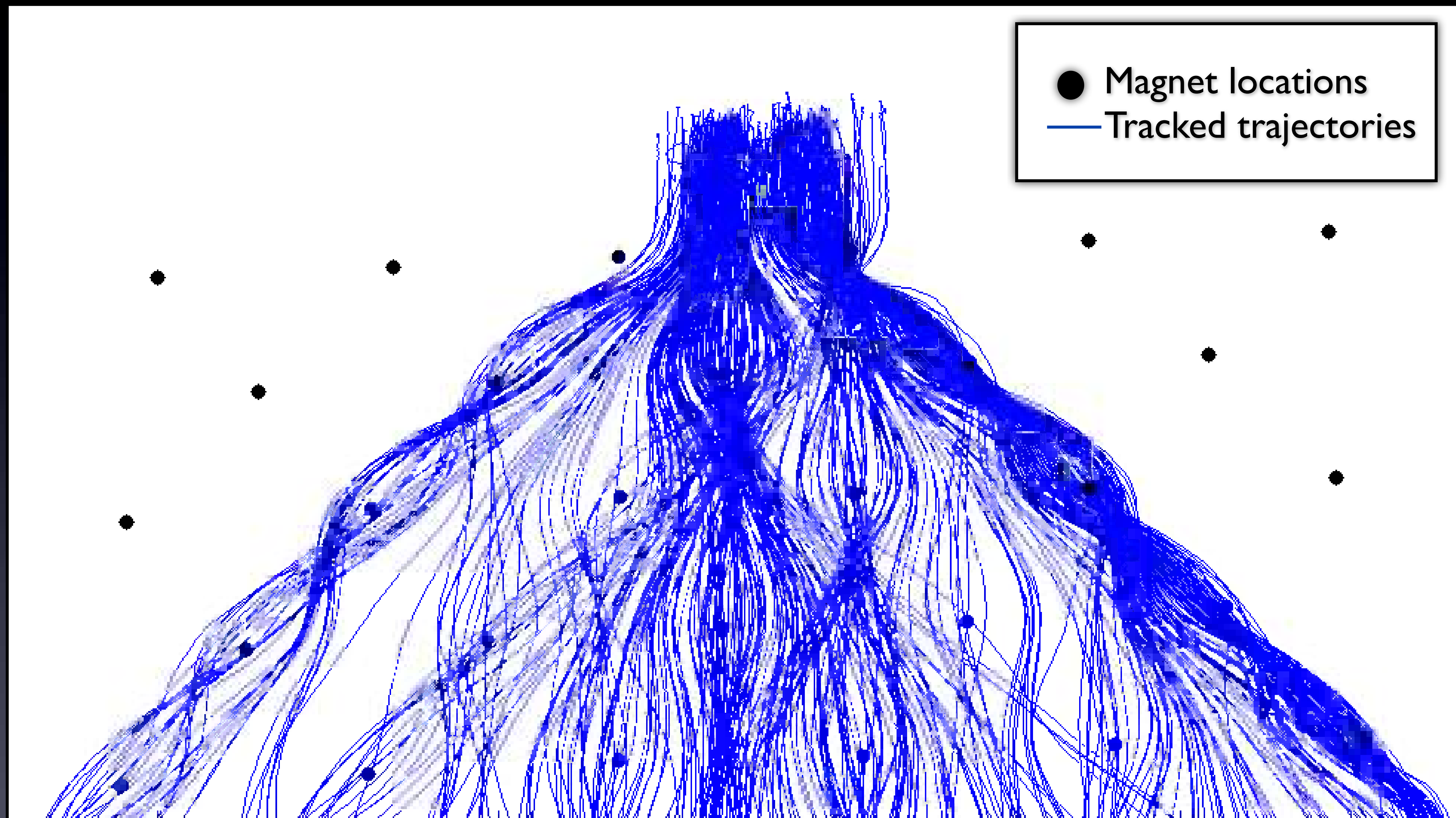
Raw Footage - 200 FPS



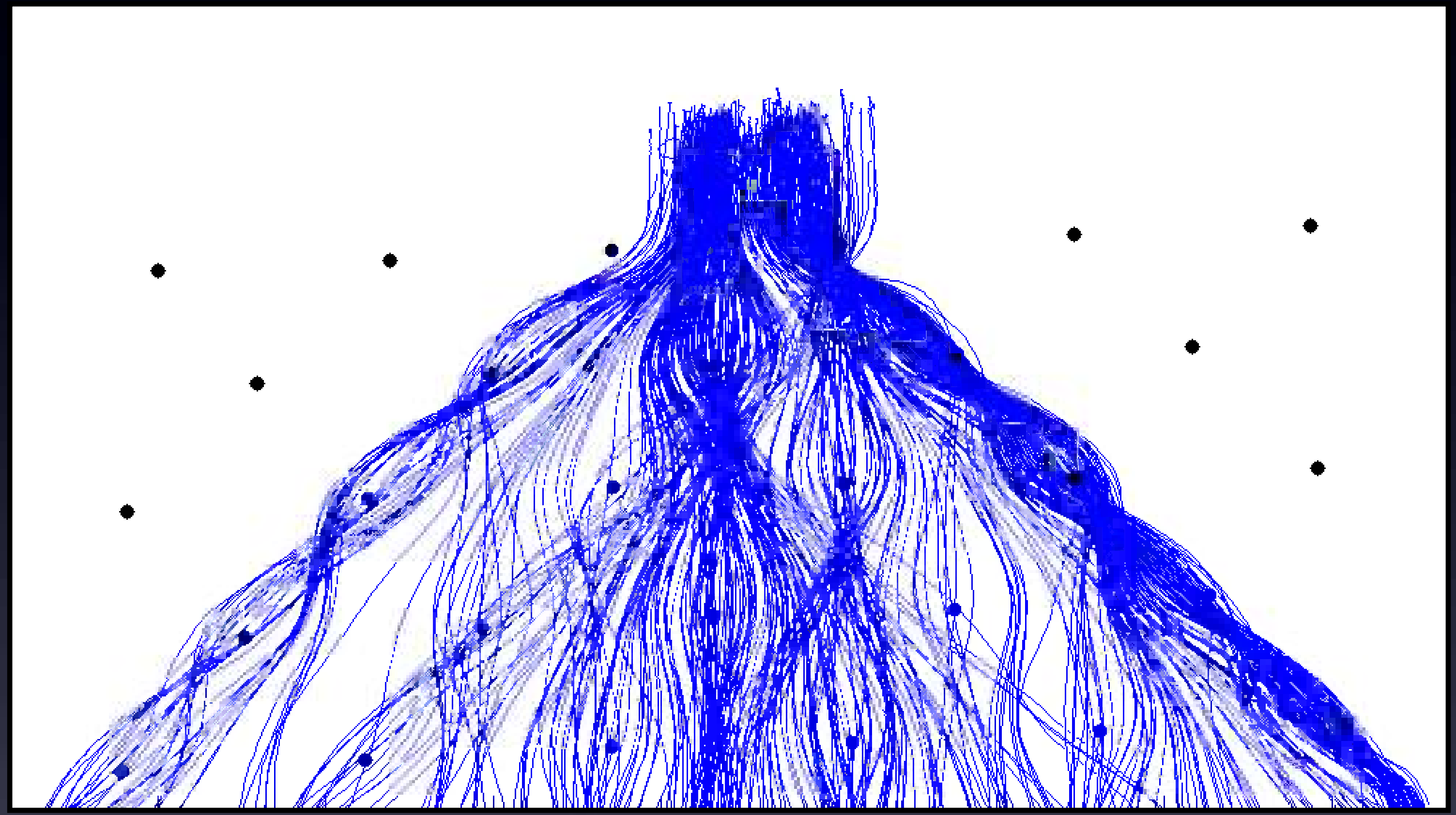




Resulting Trajectories

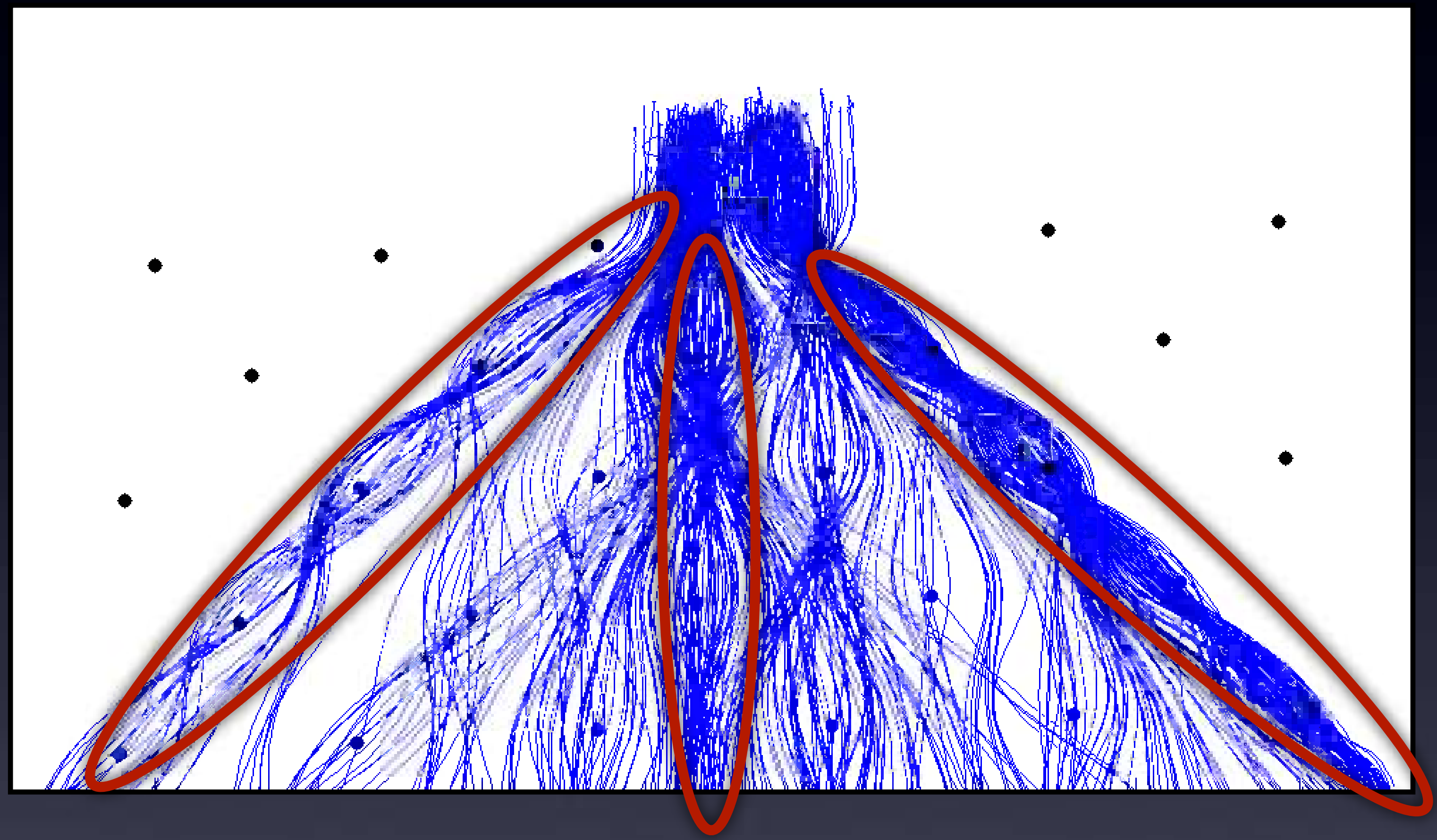


Initial Analysis



Initial Analysis

- Attracting trajectories and basins



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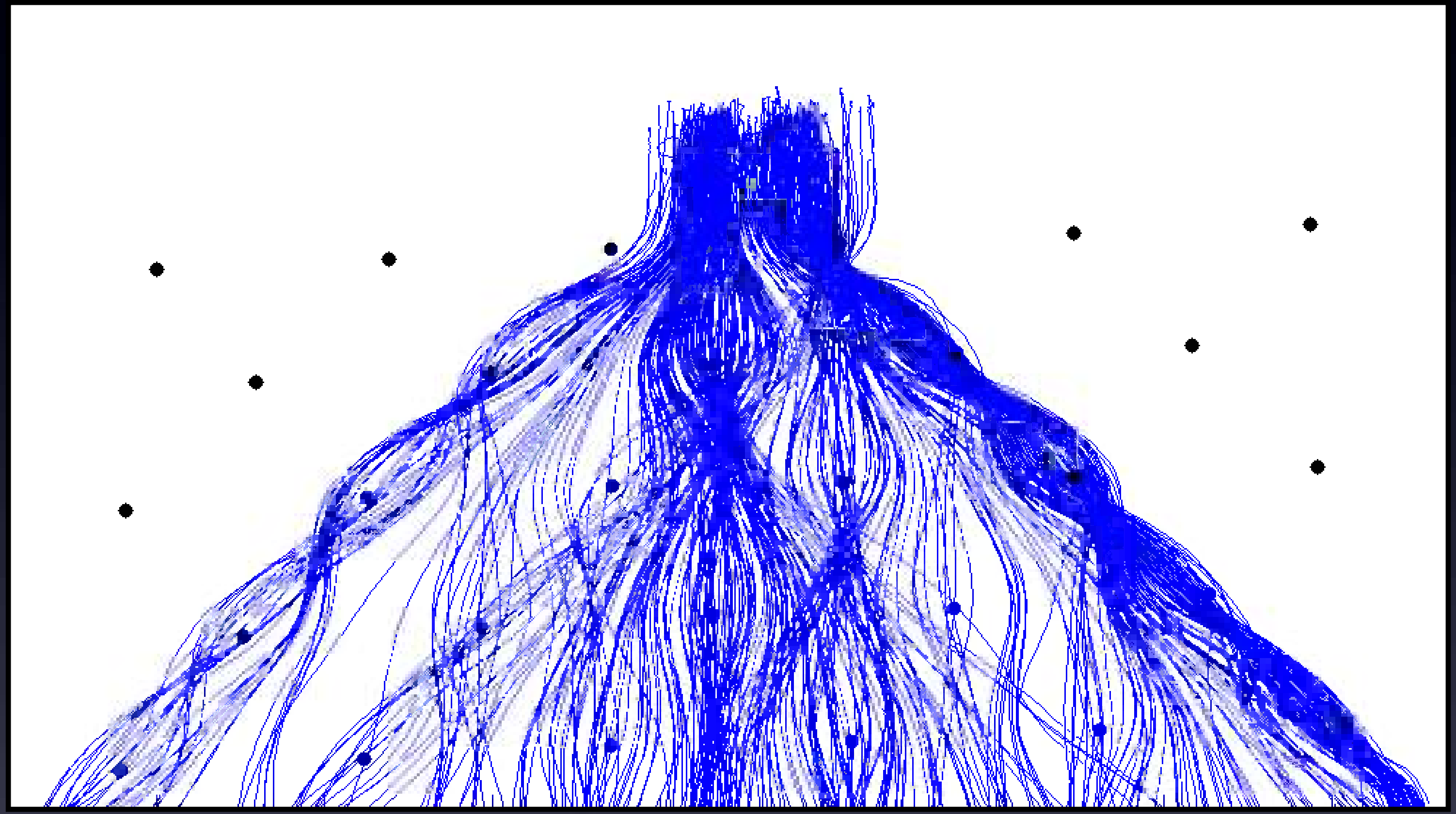
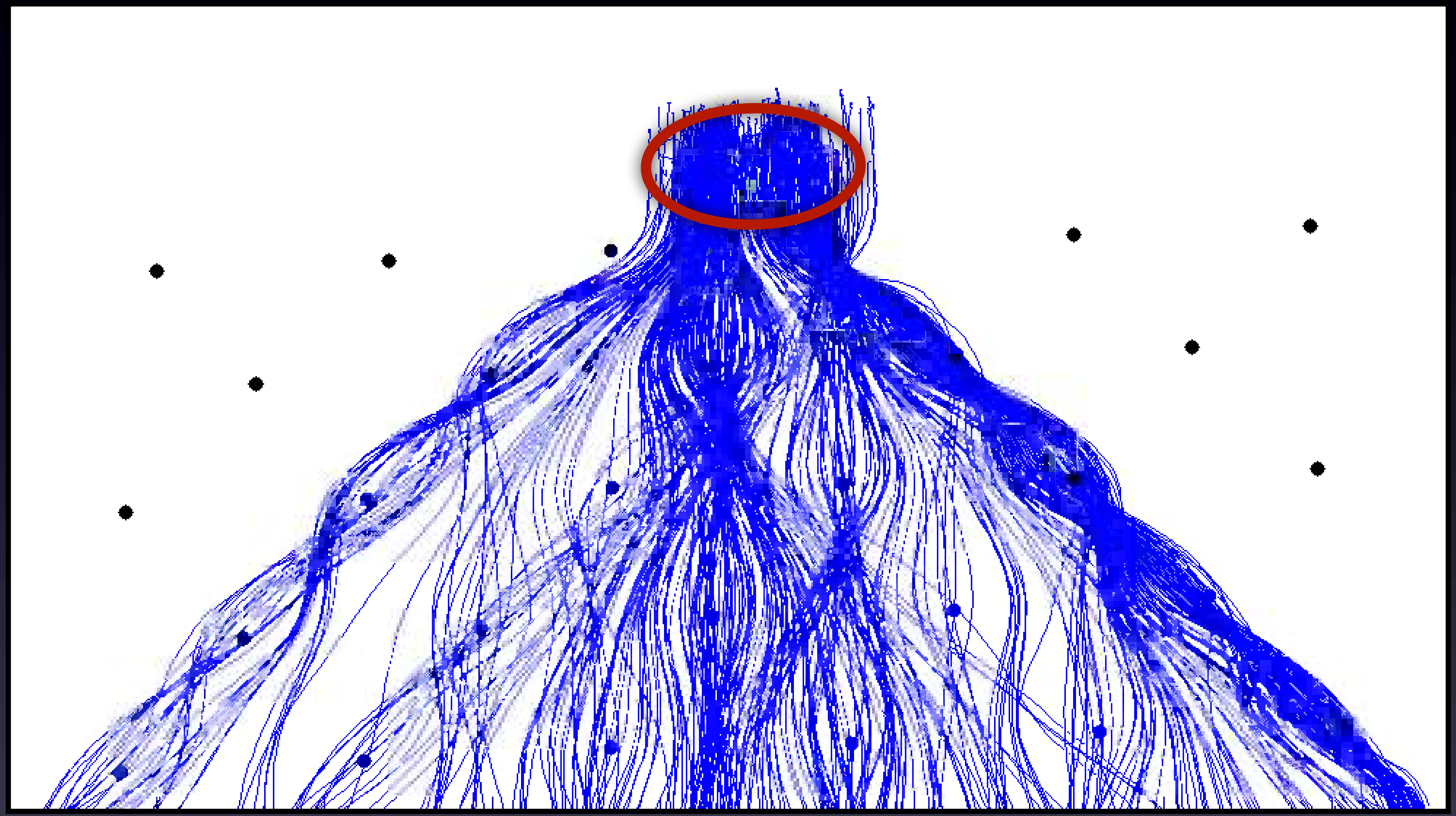


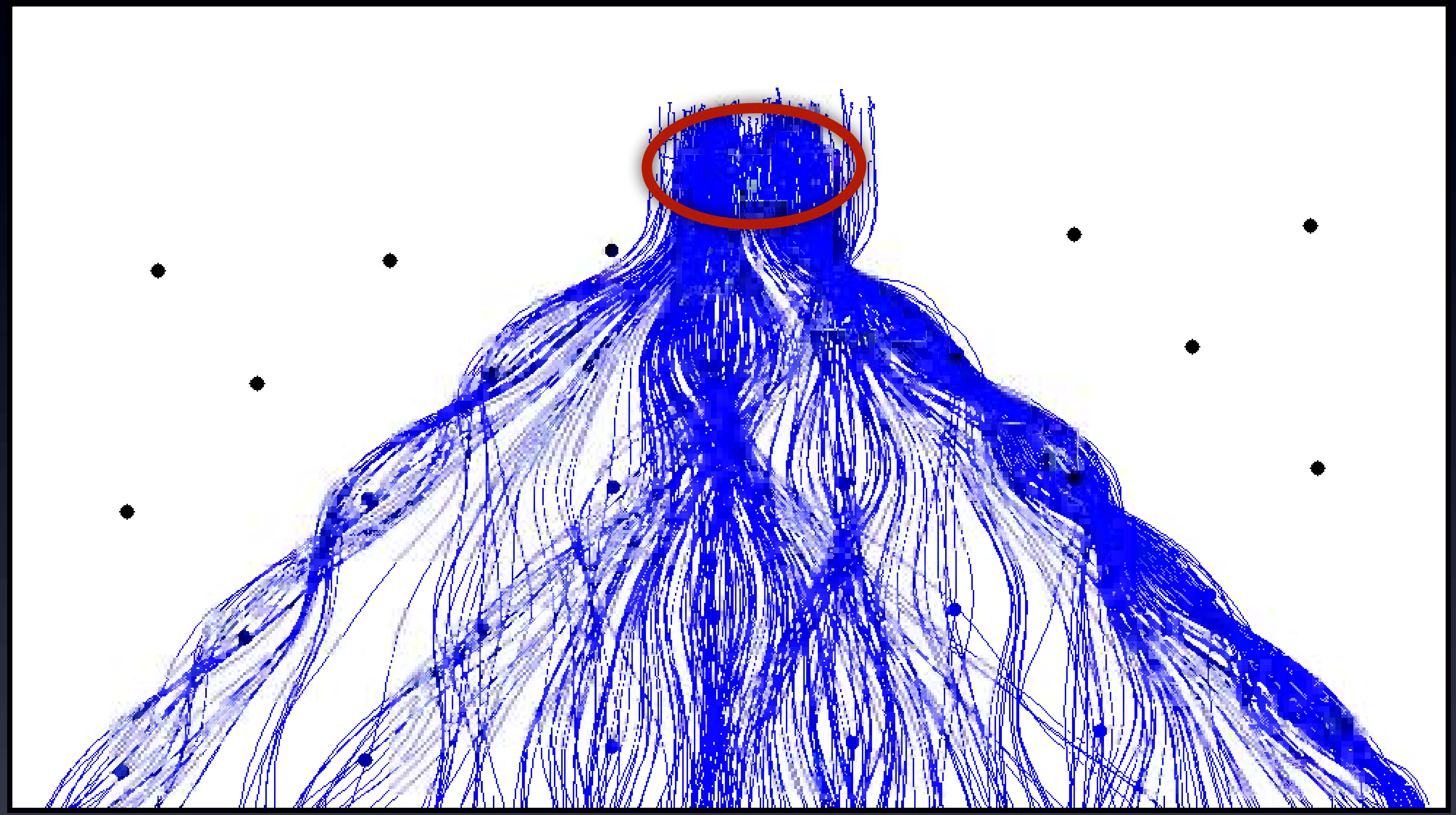
Initial Analysis

- Attracting trajectories and basins
- Sensitive dependence on initial conditions



Initial Analysis

- Attracting trajectories and basins
- Sensitive dependence on initial conditions
- Interesting structure may exist



Simulation & Data Analysis

Christopher Cordell

Simulation Introduction

- Simulation is set up to model Plinko puck movement through the magnetic Plinko board and qualitatively compare with experiment
- Code in three parts
  1) Equations of Motion file for time integration
  2) Trajectory code to run equations of motion and store trajectory
  3) Wrapper to allow for parametric study of trajectories
- Model gravitational force, friction force, total magnetic force from all magnets
- Investigate qualitative structure of basins of attraction and compare

Modeling Parameters

- Tilt angle of board ($\theta = 18°$)
- Spacing between pegs ($d = 3$ cm)
- Coefficient of kinetic friction ($\mu = 0.15$)
- Acceleration due to gravity ($g = 9.81$ m/s2)
- Depth of magnets below surface (zmag = 0.5 cm)
- Coefficient of magnetic force ($k = -0.005$)
  - Code set up such that positive k = repelling, negative k = attracting

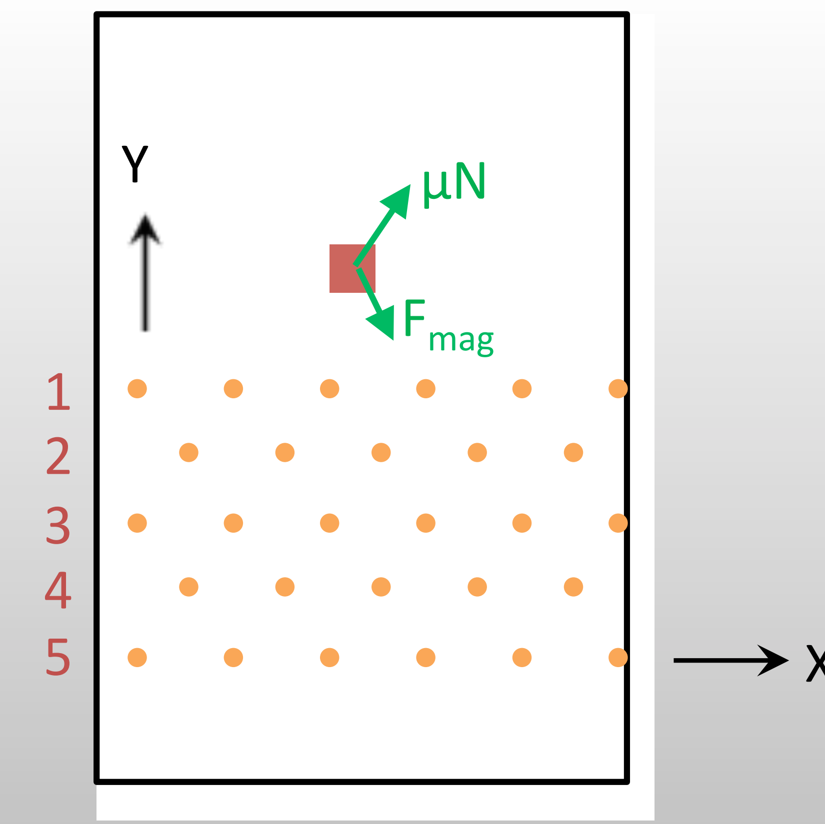
Modeling Assumptions

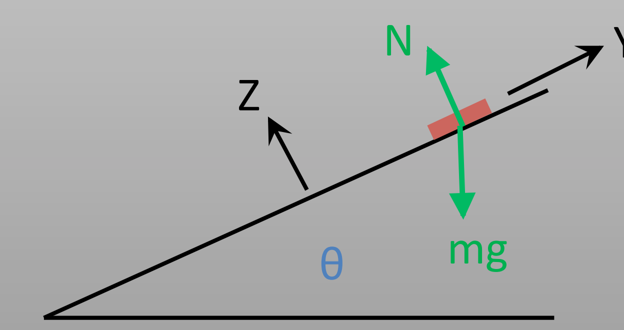
- Assume magnetic force varies as k/r^2, tune coefficient for qualitatively correct trajectories (settle on k = -0.005 for reasonable dynamics)
- Assume small magnetic poles such that they can be considered as magnetic point charges (supports k/r^2 force model)
- Rather than try to model static friction effects, assume puck is always undergoing kinetic friction (friction force is zero if puck is at rest)
- Perfectly equilateral triangle spacing of pegs (board is not exactly equilateral for actual experiment)

Equations of Motion

- Integrated in plane of board
- Gravity: $\vec{a}_g = [0, -g\sin\theta]$
- Friction: $\vec{a}_f = -\mu g\cos\theta\left[\frac{v_x}{V}, \frac{v_y}{V}\right]$
- Magnets: $\vec{R}_i = [x - X_{mag,i}, y - Y_{mag,i}]$ $\quad \vec{a}_m = \sum_{i=1}^{28}\left(\frac{k}{R_i^2}\right)\frac{\vec{R}_i}{R_i}$
- Full Equations of Motion:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ a_{g,x} + a_{f,x} + a_{m,x} \\ a_{g,y} + a_{f,y} + a_{m,y} \end{bmatrix}$$

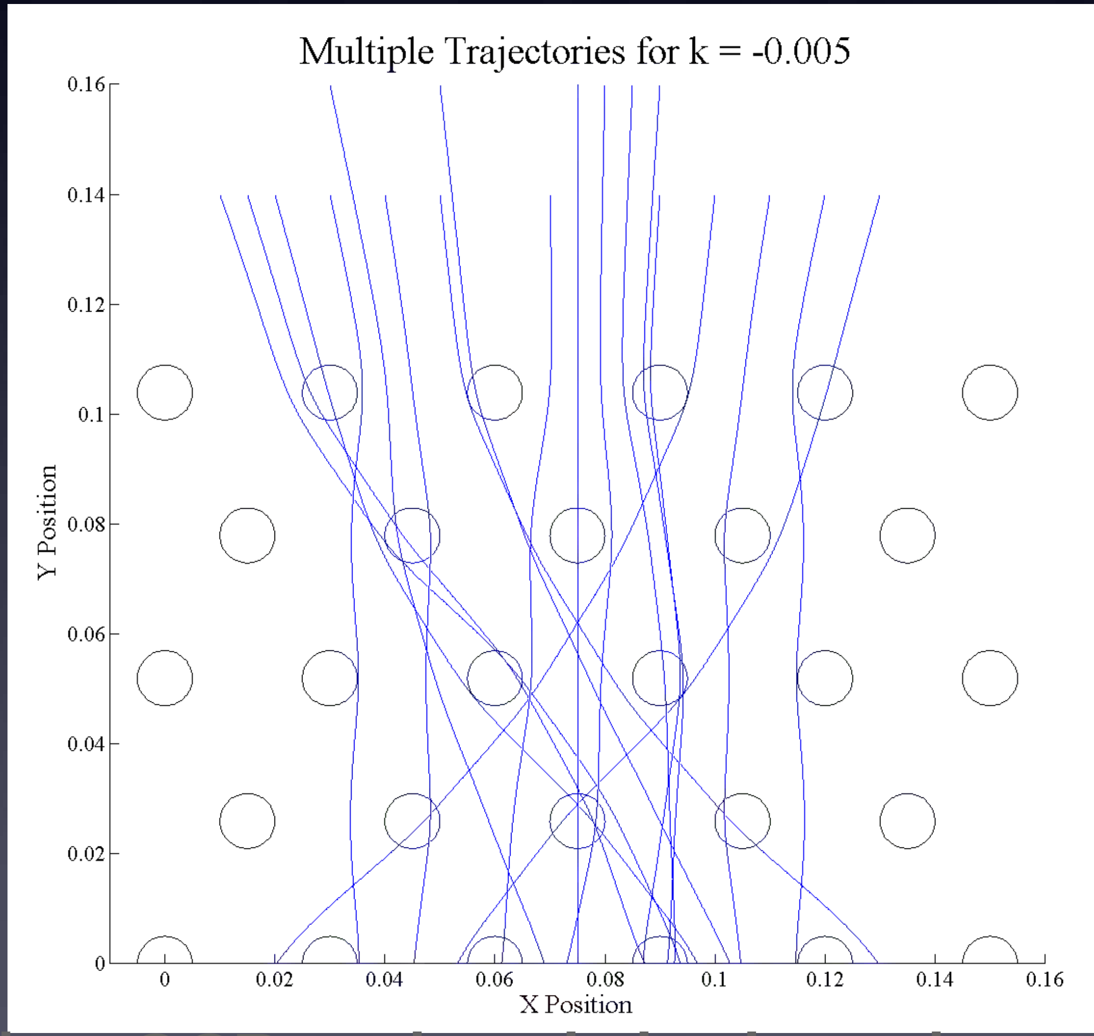


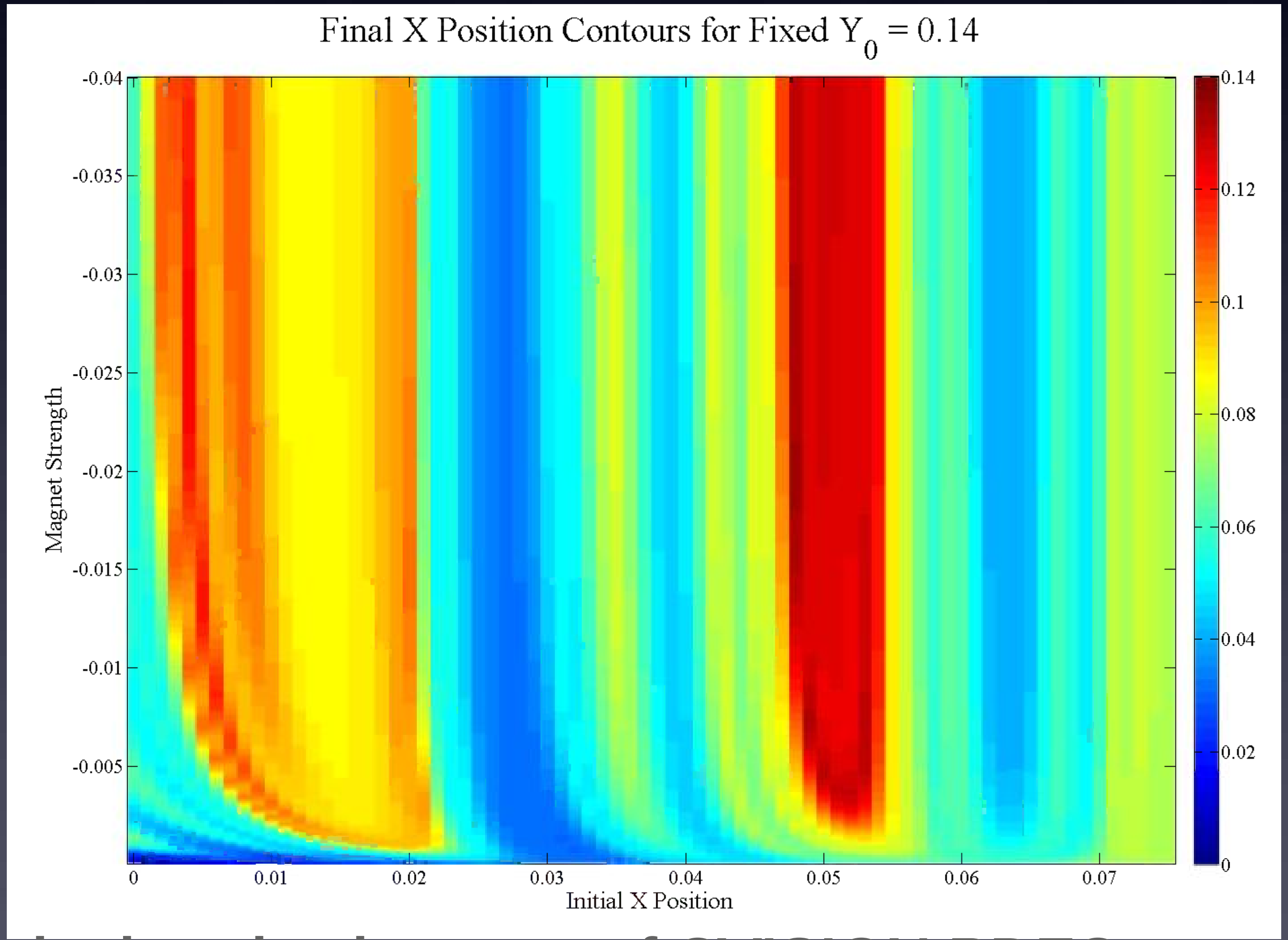


Free body representation of the system

Modeling Magnetic Force as k/r^2

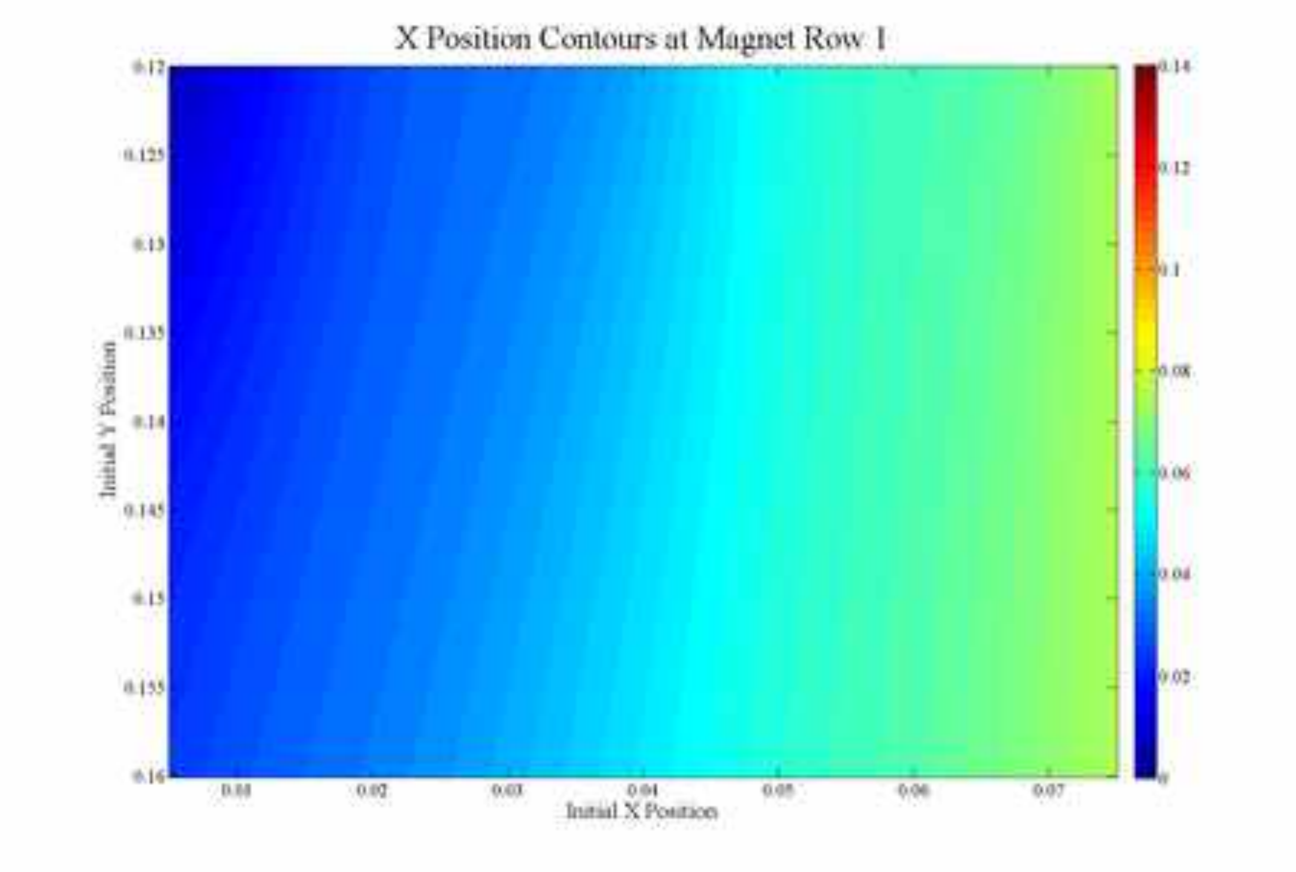
- All trajectories reach the bottom of the magnet field
- Magnetic saturation occurs once the magnetic force becomes the dominant factor in the trajectory
  - Increasing the magnet strength does not significantly alter the trajectories
  - All magnets scale at same rate, so net acceleration is unaffected

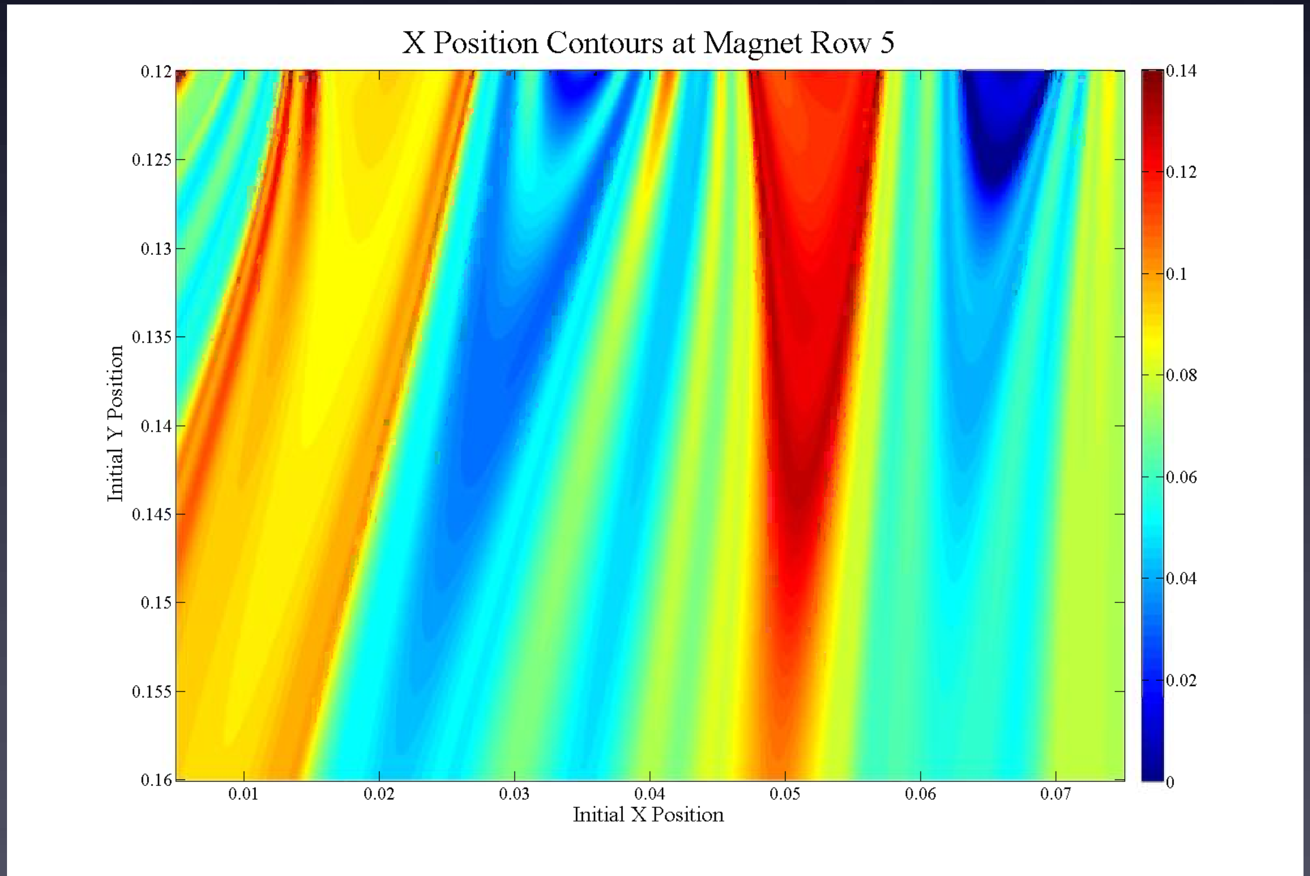




Half Plane Mapping

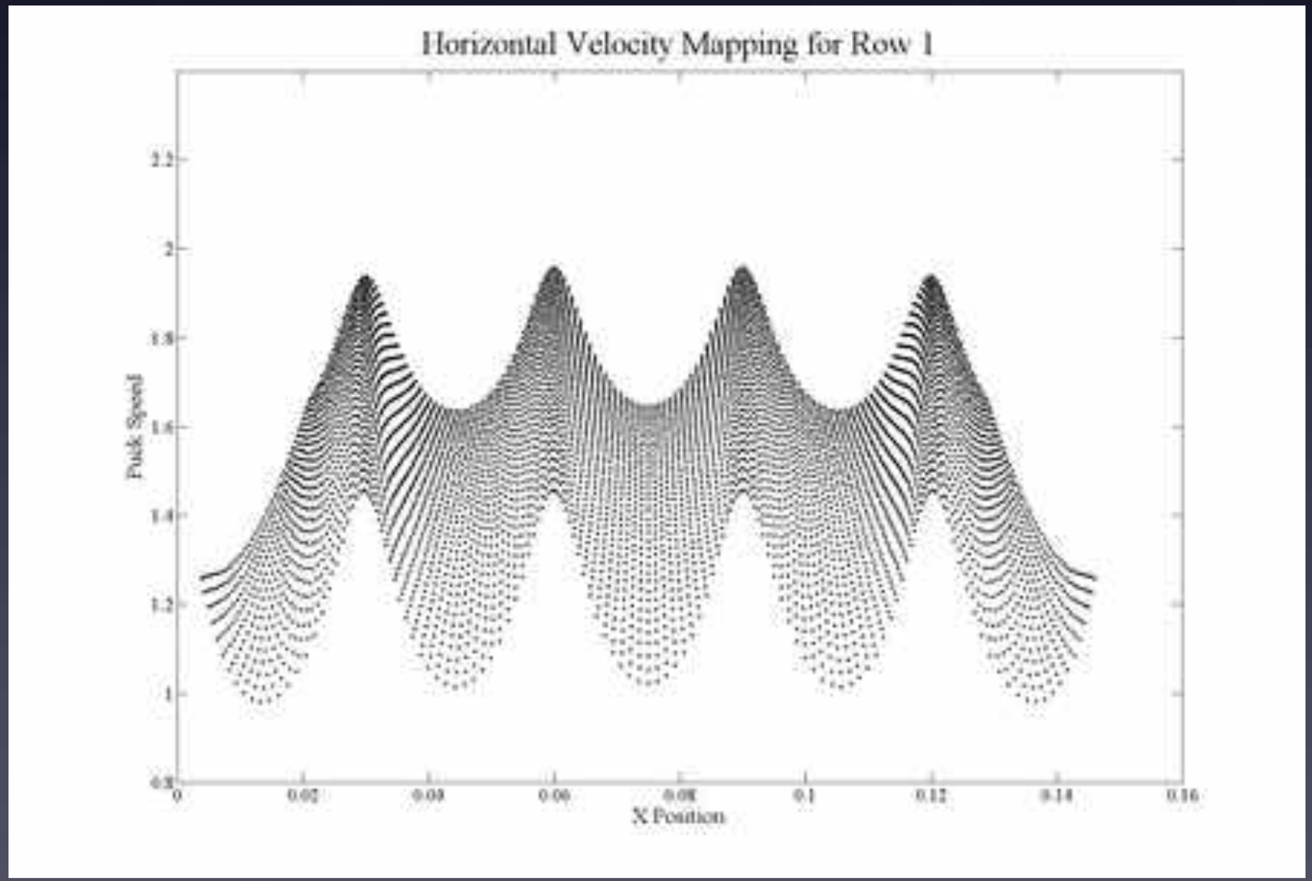
- Mapping of X position as puck passes each row of magnets
  - $X_0 \in [0.000, 0.075]$
  - $Y_0 \in [0.12, 0.16]$
- Clear basins of attraction appear
- Sensitive dependence on initial conditions in many regions

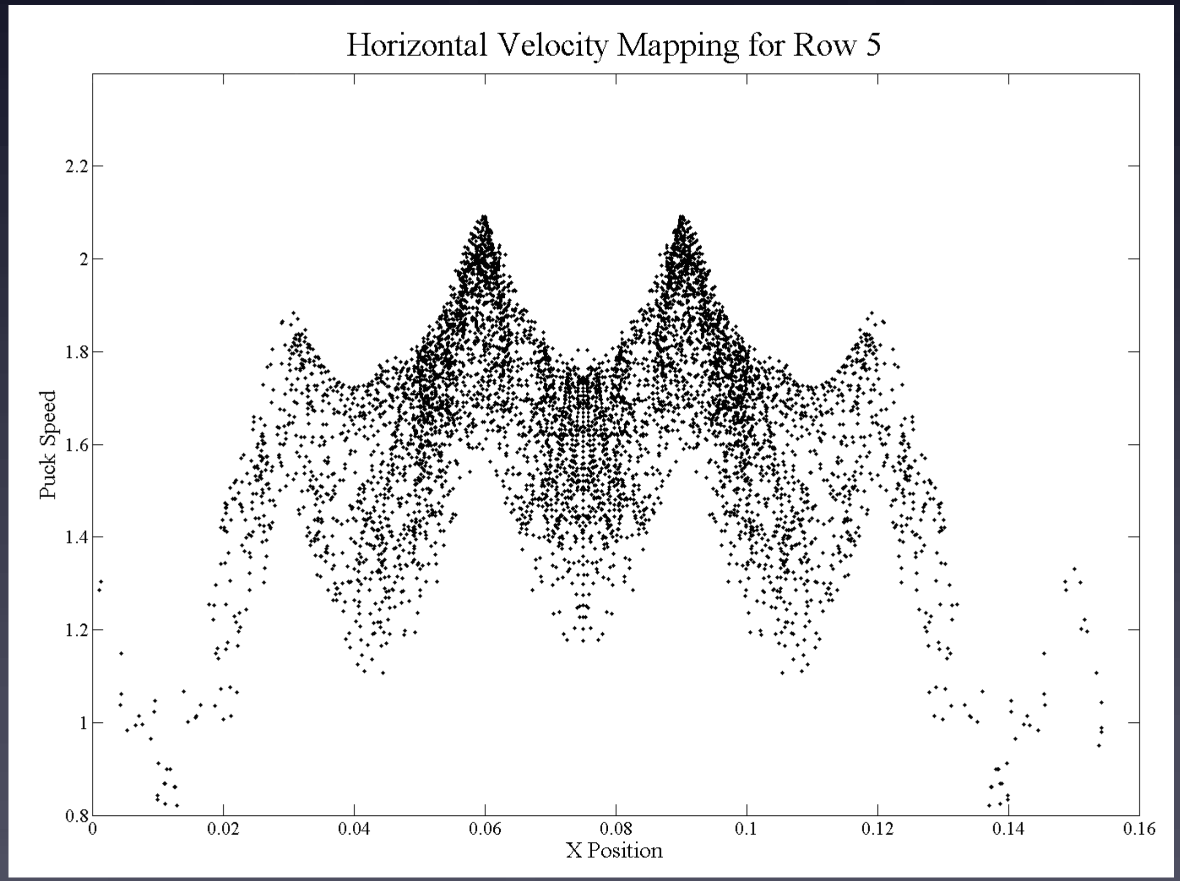




Mapping Speed at Each Row

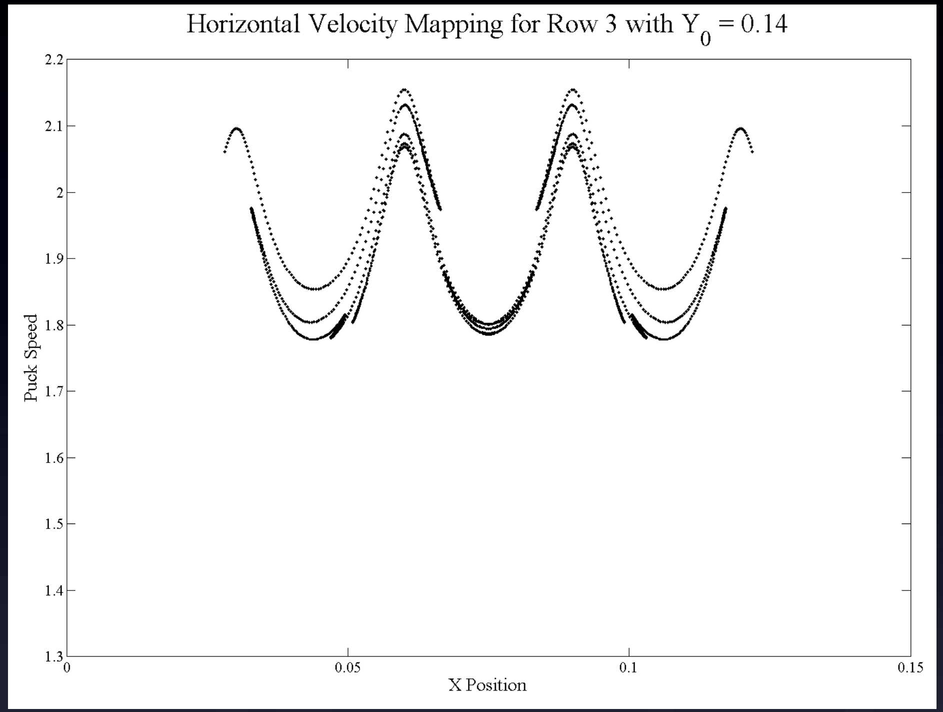
- Track puck speed as it passes each row of magnets
  - $X_0 \in [0.000, 0.150]$
  - $Y_0 \in [0.12, 0.16]$
- Plot puck speed versus X position for various trajectories to determine if there is an attractor similar to the ski-slope problem

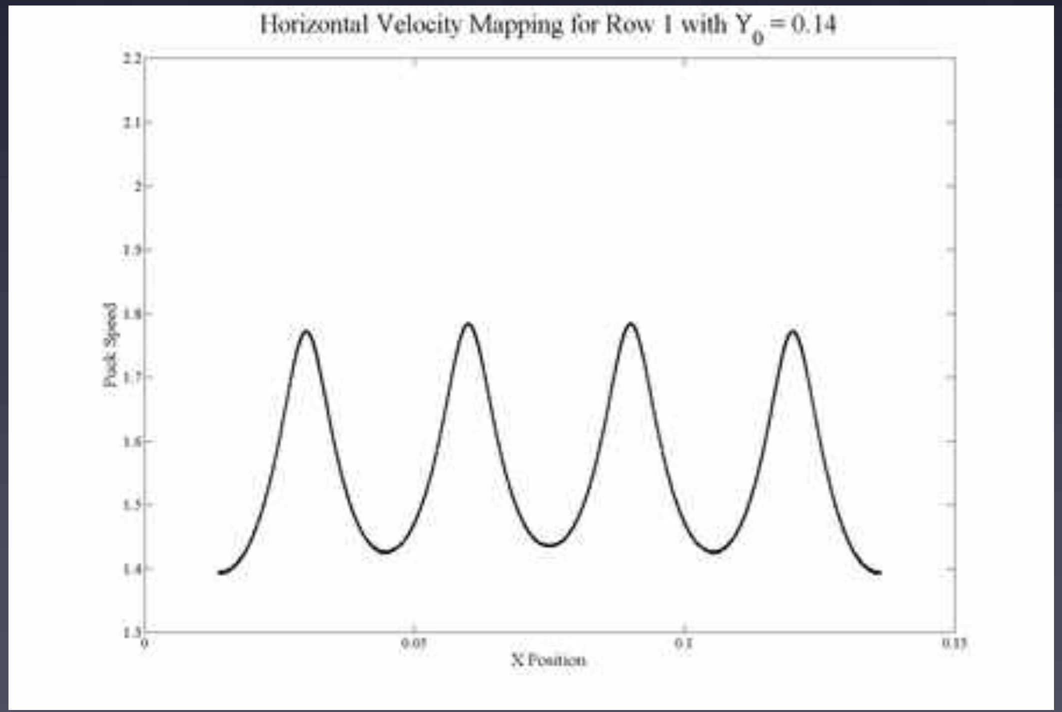


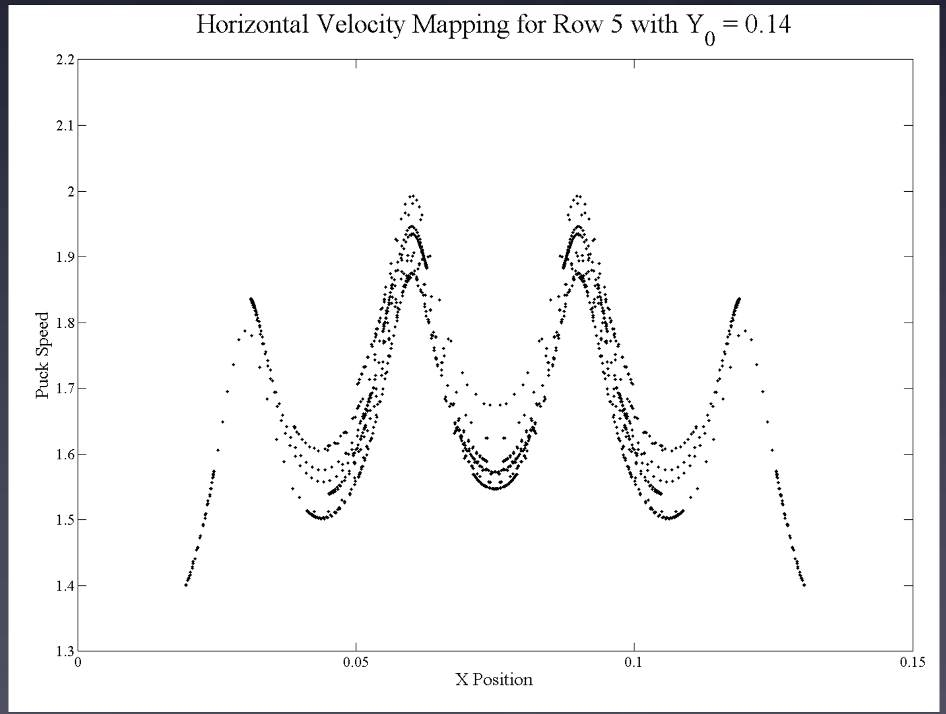


Mapping Speed at Each Row

- Fix starting height, increase resolution of X_0 spacing
  - $X_0 \in [0.000, 0.150]$
  - $Y_0 = 0.14$
- Folding occurs of map as puck descends through magnet field

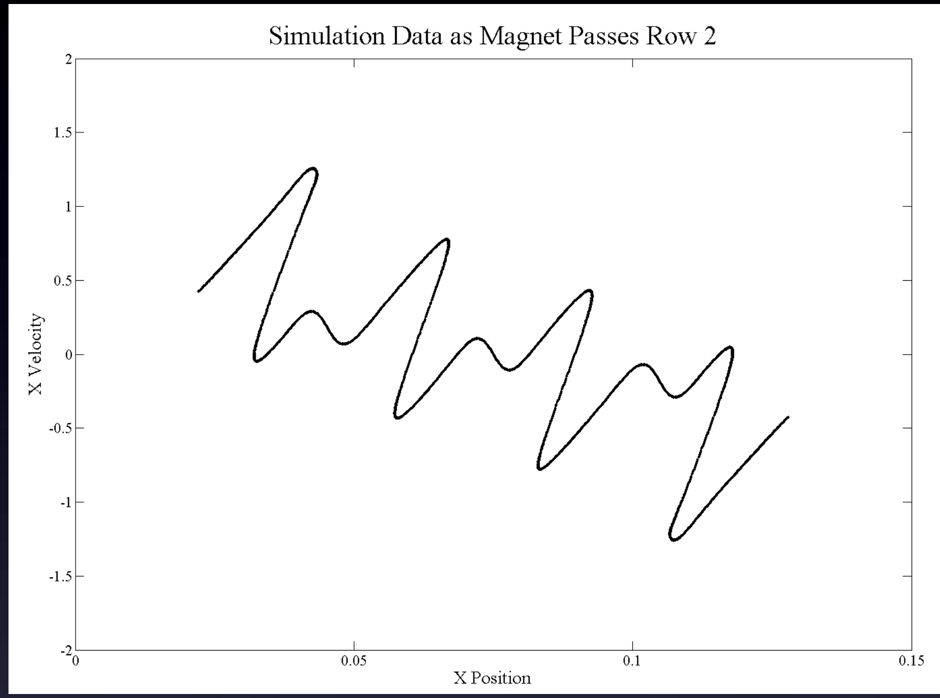


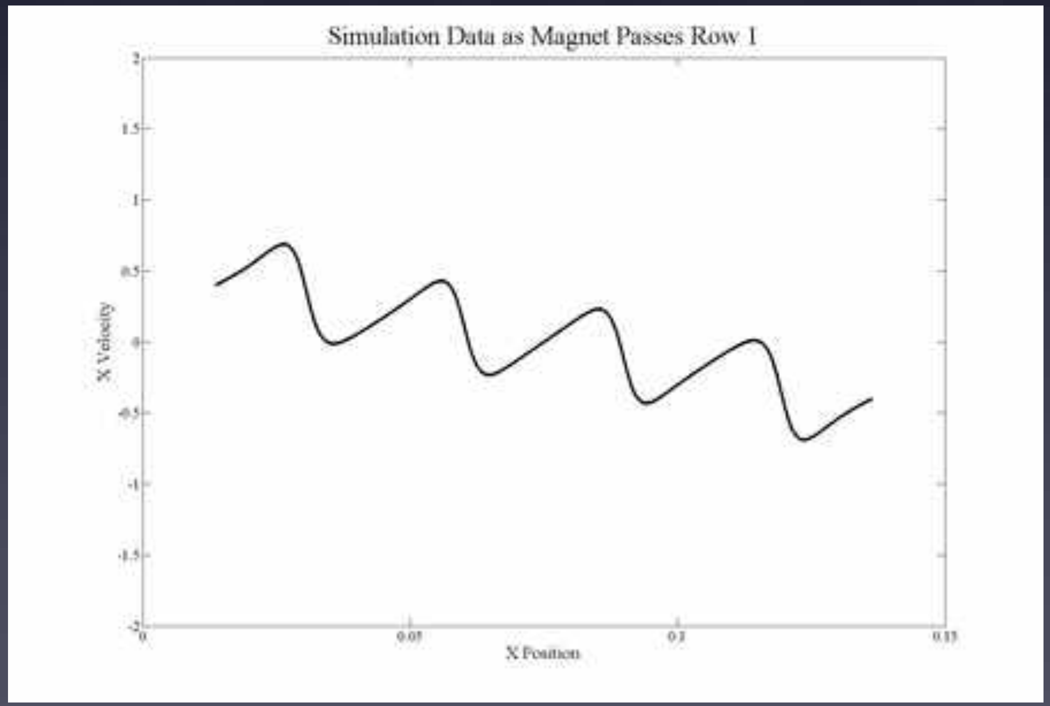


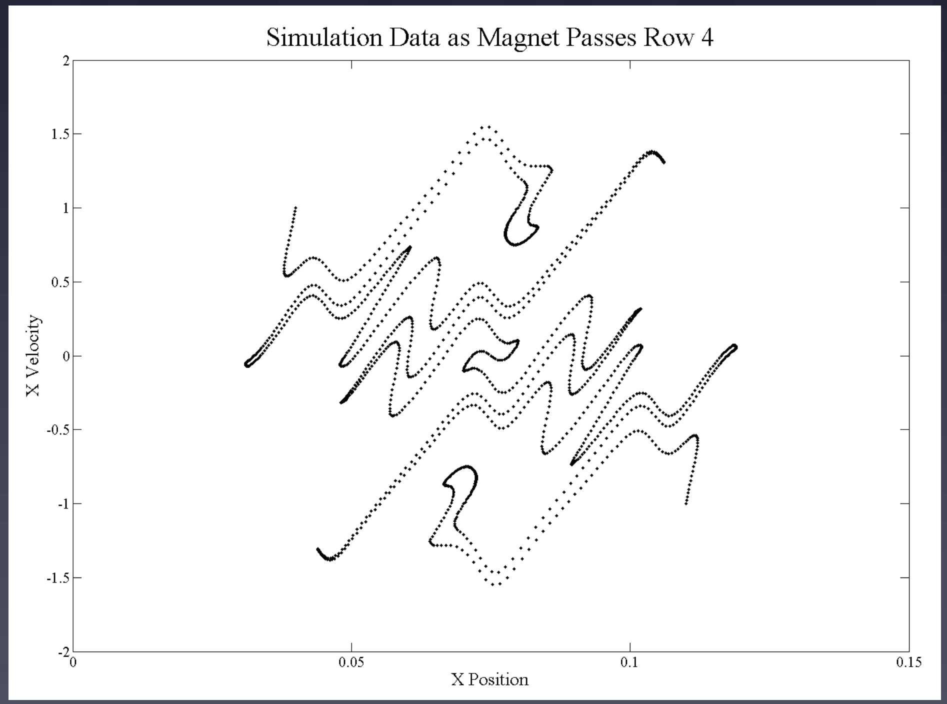


Mapping Cross-Slope Speed

- Instead of looking at how total speed varies, only look at variation in the horizontal component
- Folding of map is more evident

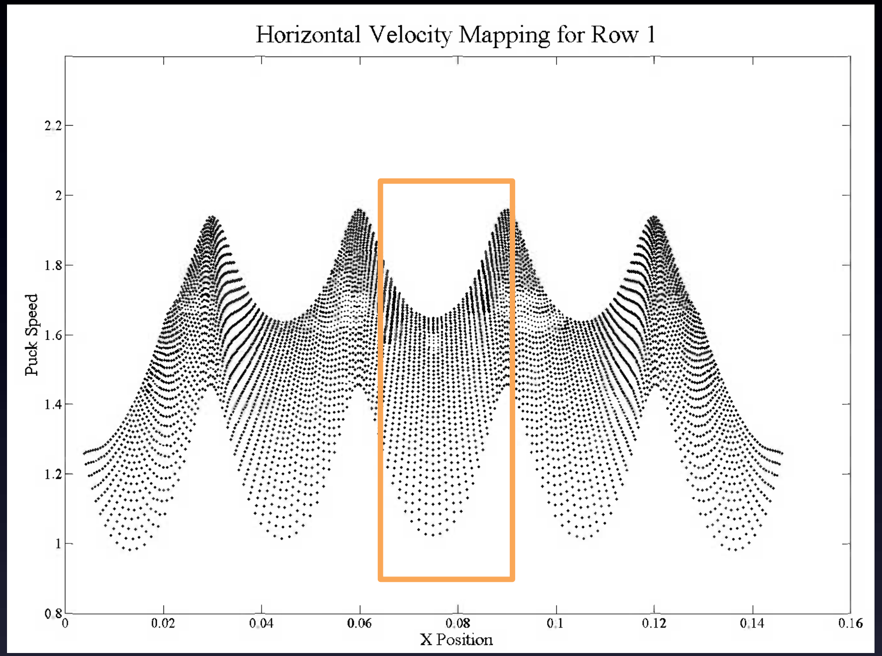


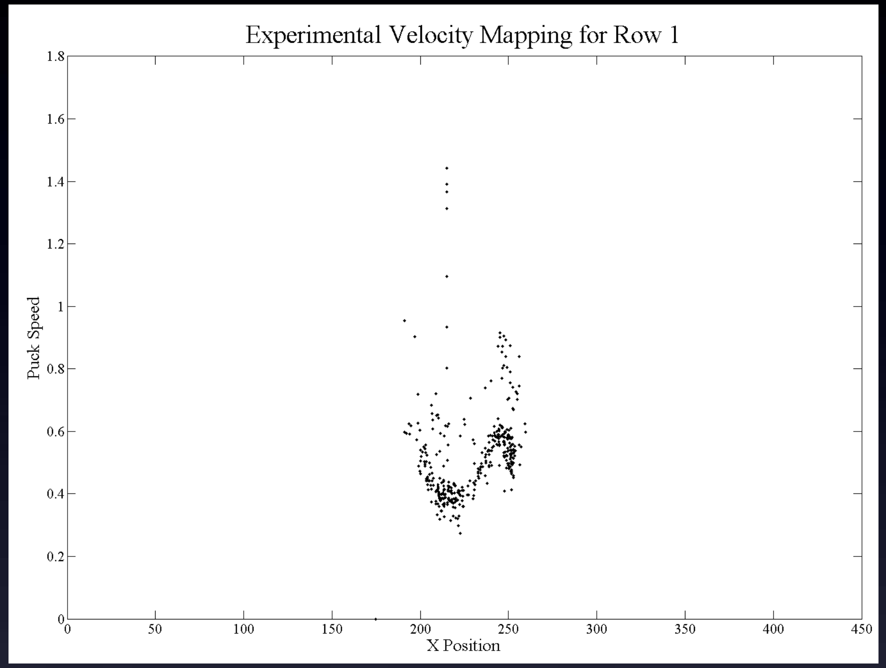


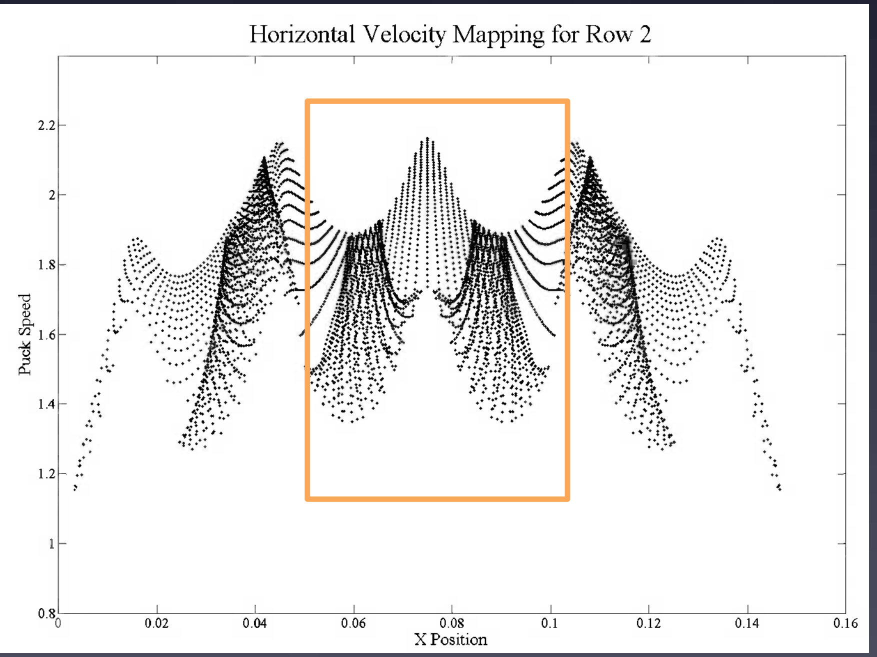


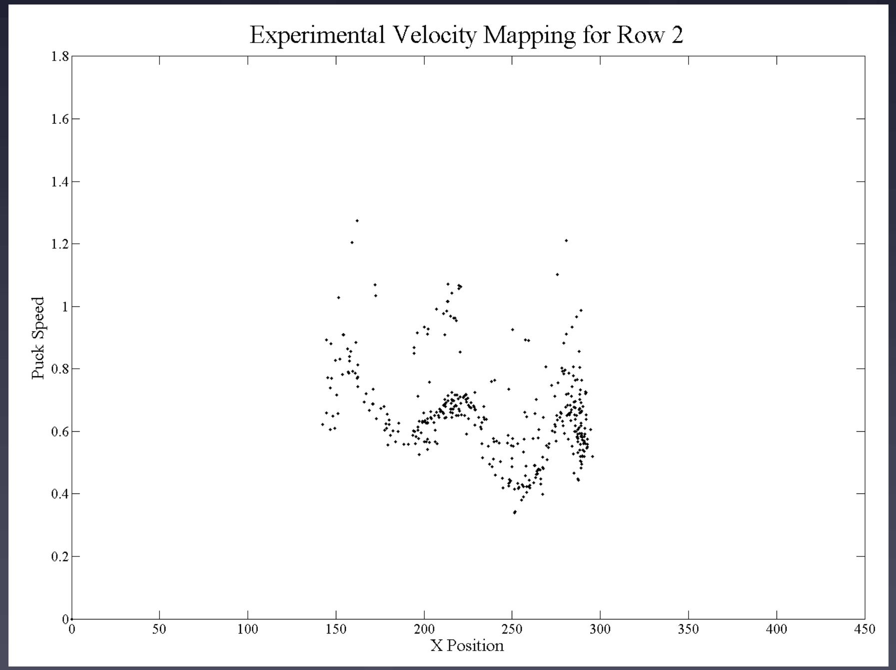
Experimental Comparison

Total Velocity vs Cross-Slope Location







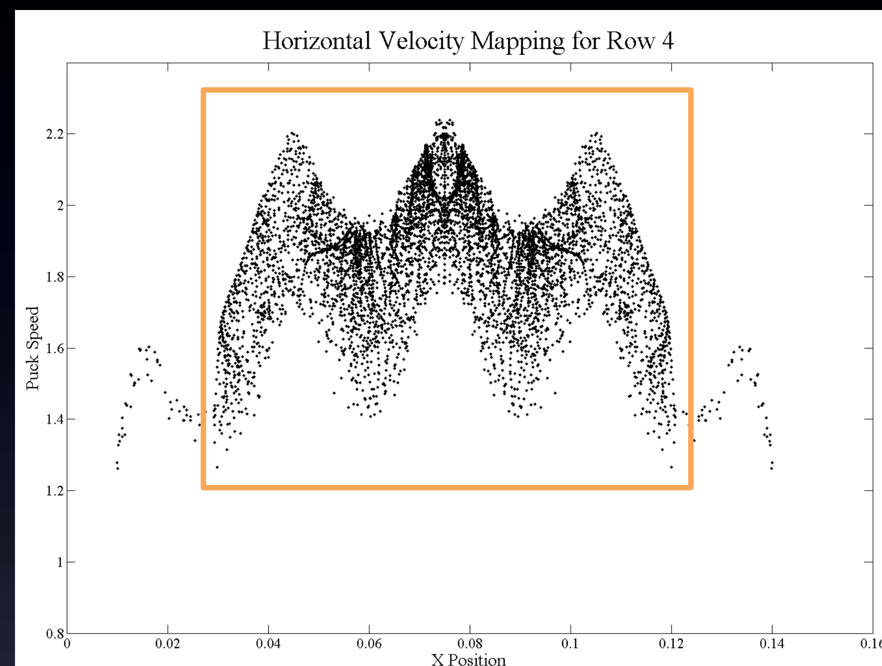


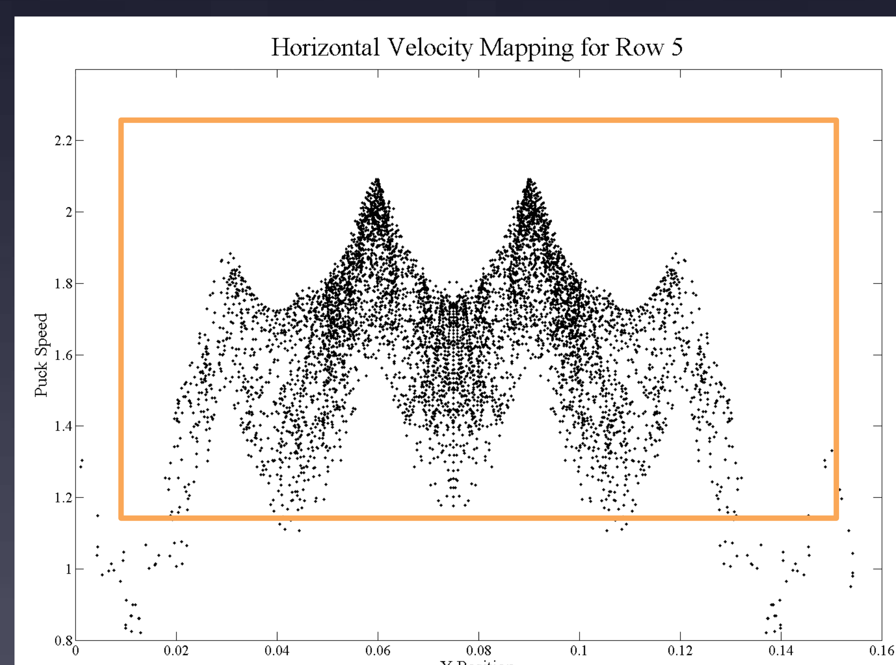
Speed (m/s) vs Location (m)

Speed (px/s) vs Location (px)

Experimental Comparison

Total Velocity vs Cross-Slope Location





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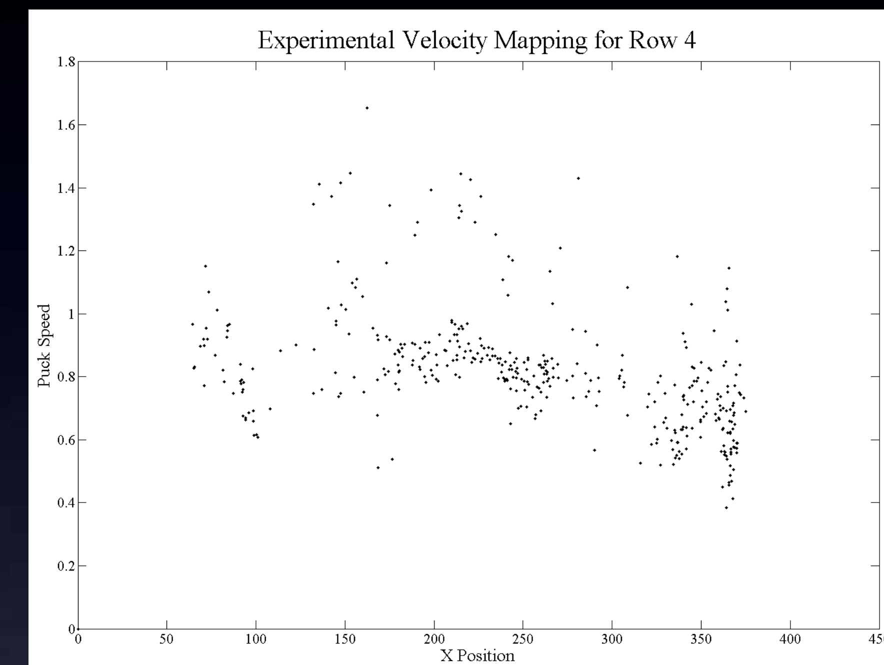
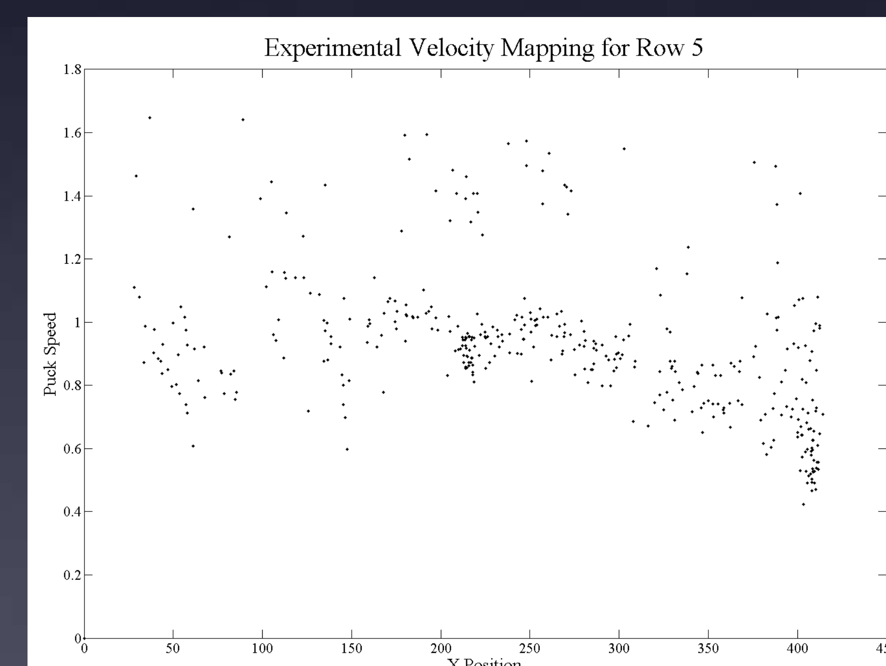
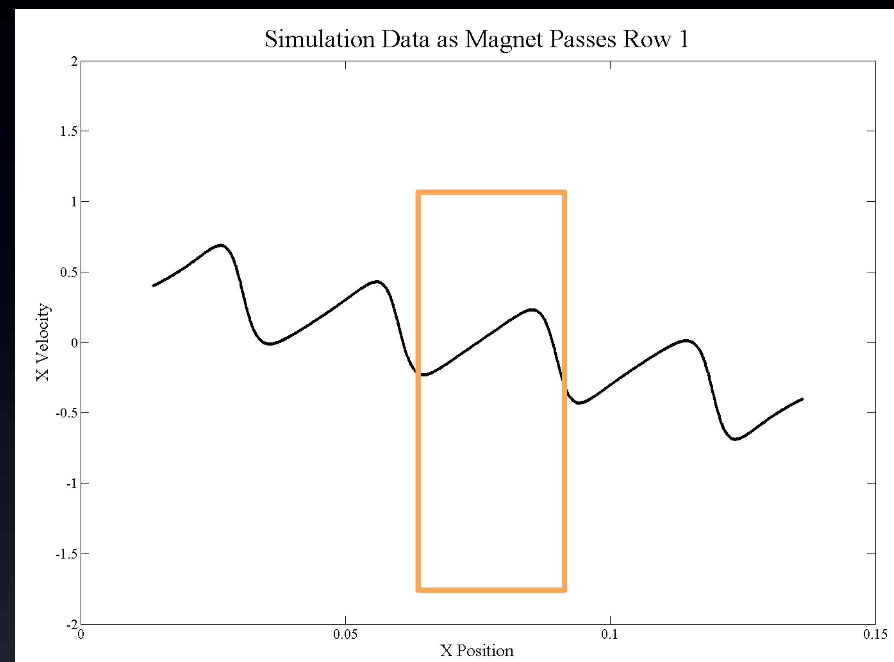


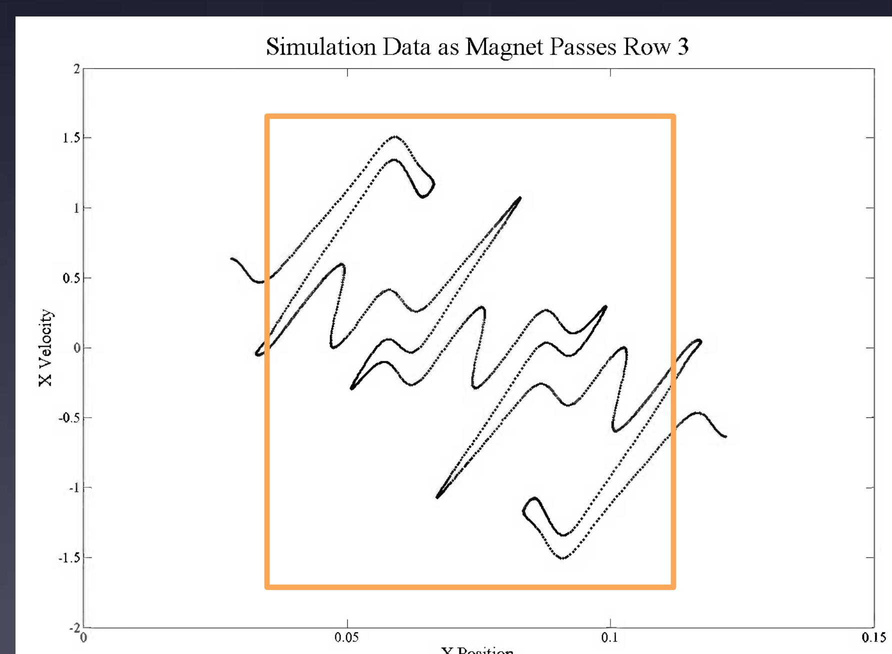


Speed (px/s) vs Location (px)

Experimental Comparison

Cross-Slope Velocity vs Cross-Slope Location





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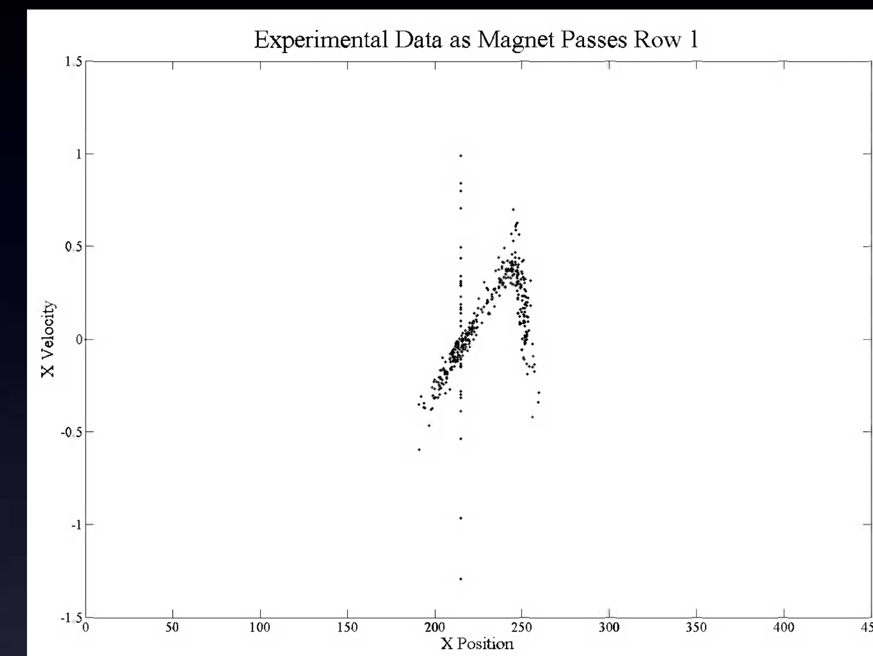
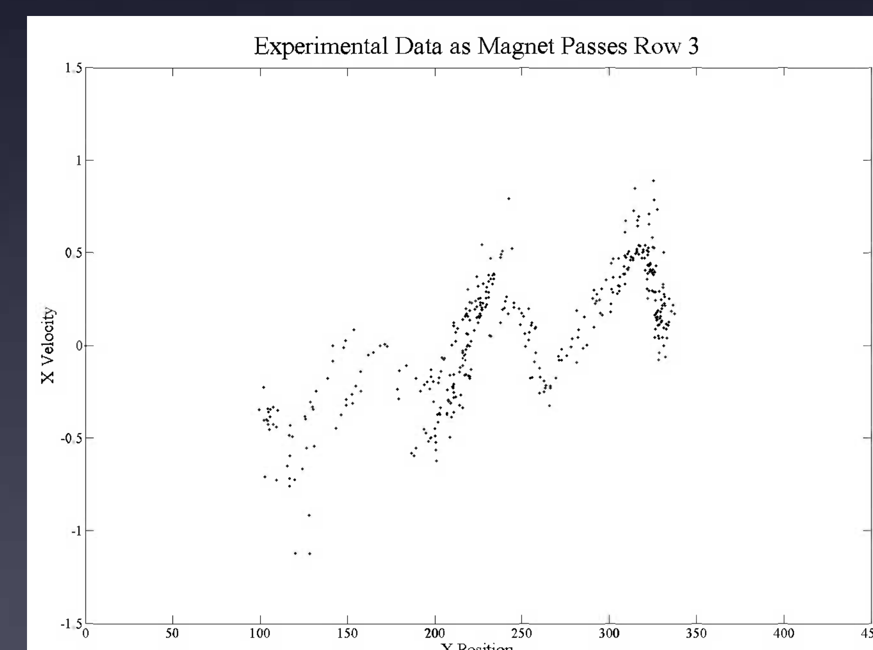




Speed (px/s) vs Location (px)

Concluding Remarks

Andrew Hardin

Conclusions

- A Plinko-like, square lattice of magnets demonstrates ski slope chaos
  - Acts as a surrogate model suitable for physical experimentation
- Clear basins of attraction exist in both the mathematical and physical systems
- Velocity and especially cross-slope velocity show chaotic folding as the puck moves through the lattice

Future Research

- Study the effects of repulsion and compare to the effects of attraction
- Many more runs from a constant altitude to clearly demonstrate chaotic folding of velocity in the experimental data
- Further evaluation of the magnetic force constant for quantitative comparison with experiment
- Examine angular variation of the board and its effect on the formation and nature of chaotic structures

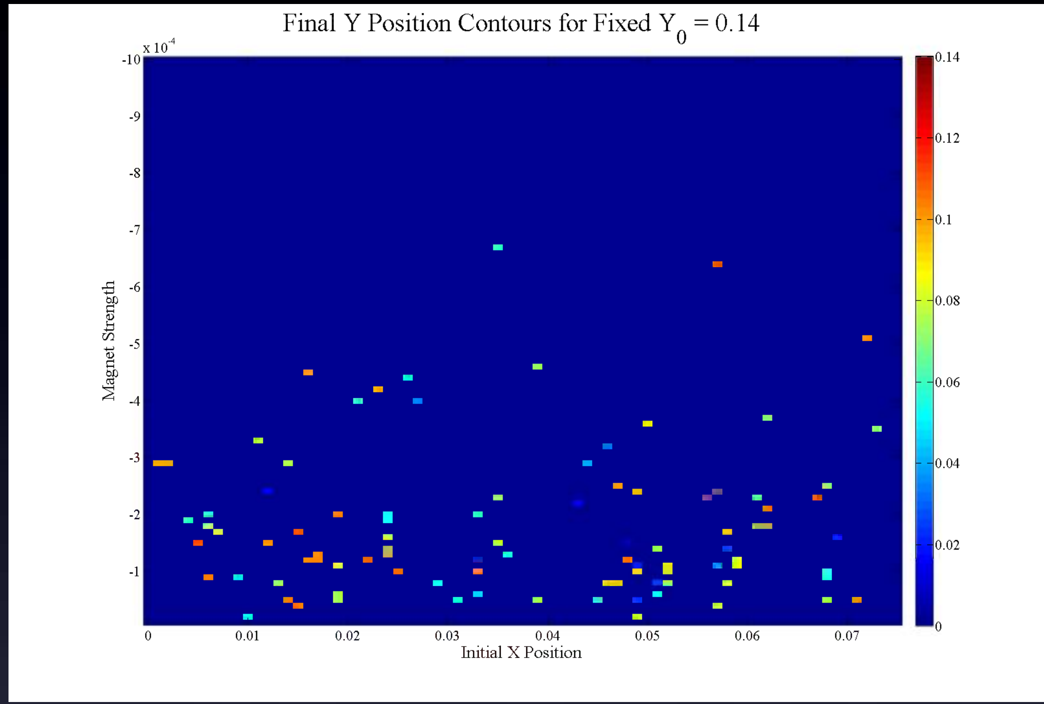
Acknowledgements

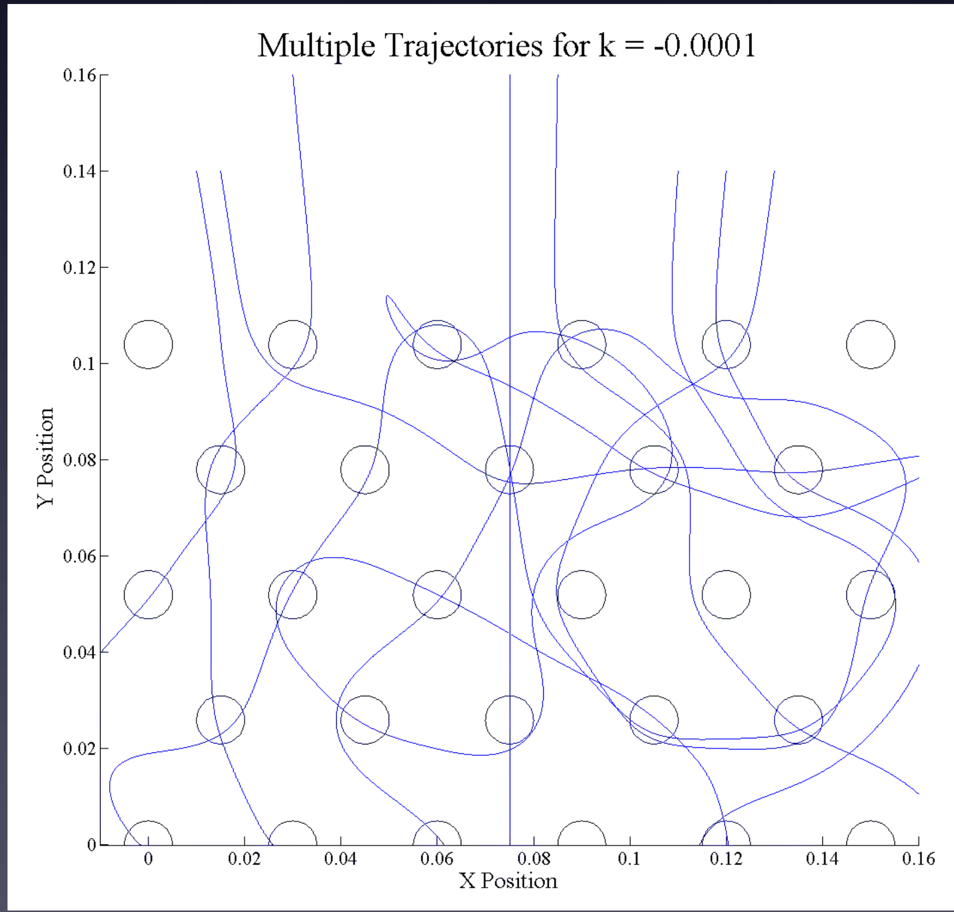
- Daniel Goldman, Professor
- Nick Gravish, Teaching Assistant
- Plinko video courtesy of CBS/Mark Goodson Productions, on the web at http://www.youtube.com/watch?v=uz8b3_jXeek

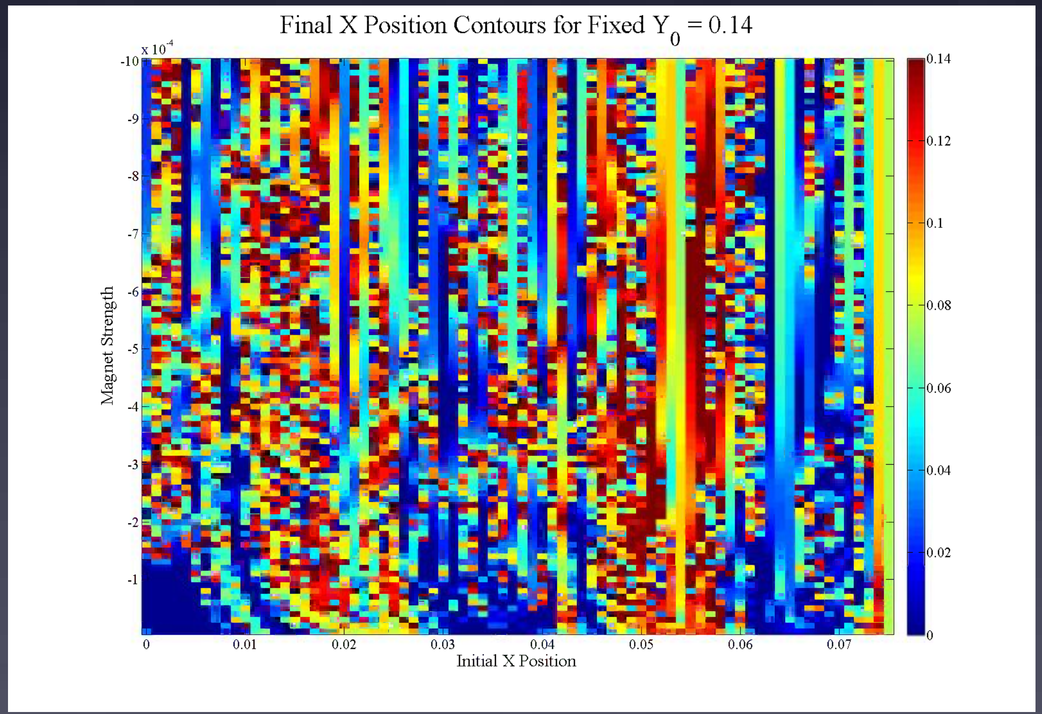
Appendix Slides

Modeling Magnet Force as k/r^3

- k/r^3 does not seem to provide qualitatively similar trajectories
- Not all trajectories reach the bottom of the magnet field







Fractal Structure Investigation

- Noted that there appeared to be a similarity region around $X_0 = 0.04$
- Ran a finer solution around this point to investigate the possible fractal structure
- No fractal structure is shown to exist at this level of resolution